

ON THE STRONG HOMOTOPY ASSOCIATIVE ALGEBRA OF A FOLIATION

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ABSTRACT. An involutive distribution C on a smooth manifold M is a Lie-algebroid acting on sections of the normal bundle TM/C . It is known that the Chevalley-Eilenberg complex associated to this representation of C possesses the structure \mathbb{X} of an L_∞ -algebroid. It is natural to interpret \mathbb{X} as the (derived) Lie algebroid of vector fields on the space \mathbb{M} of integral manifolds of C . In this paper, I show that \mathbb{X} is embedded in an A_∞ -algebra \mathbb{D} of (normal) differential operators. It is natural to interpret \mathbb{D} as the (derived) associative algebra of differential operators on \mathbb{M} . Finally, I speculate about the interpretation of \mathbb{D} as the *universal enveloping strong homotopy algebra* of \mathbb{X} .

INTRODUCTION

Let M be a finite dimensional smooth manifold and C an involutive distribution on it. In view of Fröbenius theorem the datum of C is equivalent to the datum of a foliation of M . The pair (M, C) is a finite dimensional instance of a *diffiety* (or a *D-scheme*, in the algebraic geometry language) which is a geometric object formalizing the concept of *partial differential equation*. There is a rich cohomological calculus, sometimes called *secondary calculus* [20, 21, 22], associated to a diffiety (M, C) . Secondary calculus may be interpreted to some extent as a differential calculus on the space of integral manifolds of C . All constructions of standard calculus on manifolds (vector fields, differential forms, differential operators, etc.) have a secondary analogue, i.e., a formal analogue within secondary calculus. For instance, *secondary functions* are characteristic cohomologies of C , *secondary vector fields* are characteristic cohomologies with local coefficients in normal vector fields, etc. (see the first part of [23] for a compact review of *secondary Cartan calculus*). In [24] I speculated that secondary calculus is actually a *derived differential calculus* in the sense that “*all secondary constructions come from suitable algebraic structures up to homotopy at the level of (characteristic) cochains*”. As a fundamental motivation behind this conjecture, I discussed in [24] the L_∞ -algebroid of secondary vector fields.

This is a companion paper of [24]. Here, I present a further motivation behind the above mentioned conjecture: the A_∞ -algebra of secondary (linear, scalar) differential operators. The main technical tools to show the existence of such A_∞ -algebra are homological perturbations and homotopy transfer. The strategy of the proof is the following. Let $\mathcal{D}(\overline{\Lambda})$ be the associative DG algebra of differential operators on longitudinal differential forms (i.e., differential forms along C). It projects naturally onto the DG module $\overline{\Lambda} \otimes \overline{\mathcal{D}}$ of $\overline{\Lambda}$ -valued differential operators on $C^\infty(M)$, normal to C . Actually, there are *contraction data* for $\mathcal{D}(\overline{\Lambda})$ over $\overline{\Lambda} \otimes \overline{\mathcal{D}}$ (see Subsection 1.4 for the definition of contraction data). The latter allow to induce an A_∞ -algebra structure in $\overline{\Lambda} \otimes \overline{\mathcal{D}}$ from the DG algebra structure in $\mathcal{D}(\overline{\Lambda})$. Suitable contraction data

can be constructed using purely geometric (supplementary) data as follows. First construct *Poincaré-Birkhoff-Witt* (PBW) type isomorphisms $\mathcal{D}(\overline{\Lambda}) \approx S^\bullet \text{Der} \overline{\Lambda}$ and $\overline{\Lambda} \otimes \overline{\mathcal{D}} \approx \overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$ (here $\text{Der} \overline{\Lambda}$ is the DG Lie algebroid of derivations of $\overline{\Lambda}$, and $\overline{\mathfrak{X}}$ is the module of sections of the normal bundle TM/C). Second, notice that $S^\bullet \text{Der} \overline{\Lambda}$ and $\overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$ are commutative DG algebras and there are simple contraction data for $S^\bullet \text{Der} \overline{\Lambda}$ over $\overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$. Third, use the Homological Perturbation Theorem (and the PBW isomorphisms) to construct contraction data for $\mathcal{D}(\overline{\Lambda})$ over $\overline{\Lambda} \otimes \overline{\mathcal{D}}$, from contraction data for $S^\bullet \text{Der} \overline{\Lambda}$ over $\overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$.

The paper is basically self-consistent and it is organized as follows. It is divided into three sections. In the first one, I collect the algebraic preliminaries: namely, differential operators on graded algebras, strong homotopy structures, homological perturbations and homotopy transfer. In subsection 1.5, I show how, under suitable regularity conditions (namely, the existence of a PBW type isomorphism), the universal enveloping algebra of a DG Lie algebroid contracting over a complex (\mathcal{C}, δ) , can be homotopy transferred to produce an A_∞ -algebra structure in $S^\bullet \mathcal{C}$, the symmetric algebra of \mathcal{C} (see below for details). To my knowledge, this remark appears here for the first time. In the second section, I present my main framework, which consists of some basic geometry and homological algebra of a foliation, including few not so standard aspects like (normal) differential operators on a foliated manifold. Moreover, I define a distinguished class of connections on a foliated manifold, that I call *adapted connections*. Finally, I use adapted connections to construct two suitable PBW type isomorphisms $\mathcal{D}(\overline{\Lambda}) \approx S^\bullet \text{Der} \overline{\Lambda}$ and $\overline{\Lambda} \otimes \overline{\mathcal{D}} \approx \overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$. Notice that a concept more general than an adapted connection is used in the note [14] (see also [5]) for similar purposes, in the much wider context of *Lie pairs*. Unfortunately, [14] does not contain proofs. In the third section, I collect all the constructions introduced in the preceeding sections to get the A_∞ -algebra structure in $\overline{\Lambda} \otimes \overline{\mathcal{D}}$ as outlined above. Finally, I compute the higher order components of all higher operations and, in particular, prove that they vanish from the fourth on. In the conclusions, I speculate about the interpretation of the A_∞ -algebra $\overline{\Lambda} \otimes \overline{\mathcal{D}}$ as the *universal enveloping strong homotopy algebra* of the L_∞ -algebroid $\overline{\Lambda} \otimes \overline{\mathfrak{X}}$.

I will adopt the following notations and conventions throughout the paper. The degree of a homogeneous element v in a graded vector space will be denoted by \bar{v} . However, when it appears in the exponent of a sign $(-)$, I will always omit the overbar, and write, for instance, $(-)^v$ instead of $(-)^{\bar{v}}$.

Every vector space will be over a field K of zero characteristic. If $V = \bigoplus_i V_i$ is a graded vector space, I denote by $V[1] = \bigoplus_i V[1]_i$ its suspension, i.e., the graded vector space defined by putting $V[1]_i = V_{i+1}$.

Let V_1, \dots, V_n be graded vector spaces,

$$\mathbf{v} = (v_1, \dots, v_n) \in V_1 \times \dots \times V_n,$$

and $\sigma \in S_n$ a permutation. I denote by $\chi(\sigma, \mathbf{v})$ the sign implicitly defined by

$$v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(n)} = \chi(\sigma, \mathbf{v}) v_1 \wedge \dots \wedge v_n,$$

where \wedge is the graded skew-symmetric product in the (graded) exterior algebra of $V_1 \oplus \dots \oplus V_n$.

If W is a (left) module over a graded, associative, graded commutative, unital algebra A , I denote by \odot the symmetric product in the (graded) symmetric algebra $S_A^\bullet W$ of W .

Let k_1, \dots, k_ℓ be positive integers. I denote by S_{k_1, \dots, k_ℓ} the set of (k_1, \dots, k_ℓ) -*unshuffles*, i.e., permutations σ of $\{1, \dots, k_1 + \dots + k_\ell\}$ such that

$$\sigma(k_1 + \dots + k_{i-1} + 1) < \dots < \sigma(k_1 + \dots + k_{i-1} + k_i), \quad i = 1, \dots, \ell.$$

The pairing between vectors and covectors will be denoted by $\langle \cdot, \cdot \rangle$.

If S is a set, I denote

$$S^k := \underbrace{S \times \dots \times S}_{k \text{ times}}$$

and the element $(s, \dots, s) \in S^k$ of the diagonal will be simply denoted by s^k , $s \in S$.

Now, let M be a smooth manifold. I denote by $C^\infty(M)$ the real algebra of smooth functions on M , by $\mathfrak{X}(M)$ the Lie-Rinehart algebra of vector fields on M , and by $\Lambda(M)$ the DG algebra of differential forms on M . Elements in $\mathfrak{X}(M)$ are always understood as derivations of $C^\infty(M)$. Homogeneous elements in $\Lambda(M)$ are always understood as $C^\infty(M)$ -valued, skew-symmetric, multilinear maps on $\mathfrak{X}(M)$. I denote by $d : \Lambda(M) \rightarrow \Lambda(M)$ the exterior differential. Every tensor product will be over $C^\infty(M)$, if not explicitly stated otherwise, and will be simply denoted by \otimes . I adopt the Einstein summation convention.

By a *connection* I will mean a linear connection in T^*M or, which is the same, in TM . Moreover, I will always understand the obvious extension of a connection to the whole tensor bundle $\bigoplus_{i,j} TM^{\otimes i} \otimes T^*M^{\otimes j}$. Let ∇ be a connection, \dots, z^a, \dots coordinates in M , and T a covariant tensor on M locally given by

$$T = T_{a_1 \dots a_k} dz^{a_1} \otimes \dots \otimes dz^{a_k}.$$

I denote by $\nabla_a T_{a_1 \dots a_k}$ the components of the covariant derivative ∇T of T with respect to ∇ , i.e.,

$$\nabla T = \nabla_a T_{a_1 \dots a_k} dz^a \otimes dz^{a_1} \otimes \dots \otimes dz^{a_k}.$$

Finally, the round bracket in $T_{(a_1 \dots a_k)}$ denotes symmetrization, i.e., $T_{(a_1 \dots a_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{\sigma(1) \dots \sigma(k)}$.

1. ALGEBRAIC PRELIMINARIES

1.1. Differential Operators over Graded Commutative Algebras. Let A be a graded, associative, graded commutative, unital K -algebra, and let P, Q be (left) A -modules. An element $a \in A$, define endomorphisms (multiplications by a) $P \rightarrow P$ and $Q \rightarrow Q$ which, abusing the notation, I denote again by a . Consider the graded A -linear map

$$\delta_a : \text{Hom}_K(P, Q) \rightarrow \text{Hom}_K(P, Q)$$

defined by

$$\delta_a \phi := [a, \phi] := a \circ \phi - (-)^{a\phi} \phi \circ a,$$

where $[\cdot, \cdot]$ is the graded commutator. A graded, K -linear map

$$\square : P \rightarrow Q$$

is a (linear) differential operator of order k if

$$\delta_{a_0} \delta_{a_1} \dots \delta_{a_k} \square = 0 \quad \text{for all } a_0, a_1, \dots, a_k \in A.$$

Example 1. A derivation of A is a differential operator of order 1. More generally, a derivation $\square : P \rightarrow P$ over a derivation Δ in A , i.e., an operator \square such that

$$\square(ap) = \Delta(a)p + (-)^{a\square} a\square p, \quad a \in A, \quad p \in P,$$

is a differential operator of order 1.

The left A -module of differential operators $\square : P \rightarrow Q$ of order k will be denoted by $\mathcal{D}_k(P, Q)$. Clearly, $\mathcal{D}_0(P, Q) = \text{Hom}_A(P, Q)$ and there is a chain of A -modules

$$\mathcal{D}_0(P, Q) \subset \cdots \subset \mathcal{D}_k(P, Q) \subset \mathcal{D}_{k+1}(P, Q) \subset \cdots,$$

defining a filtration in the A -module $\mathcal{D}(P, Q) := \bigcup_k \mathcal{D}_k(P, Q)$. The associated graded object $\mathcal{S}(P, Q) := \bigoplus_k \mathcal{S}_k(P, Q)$, $\mathcal{S}_k(P, Q) := \mathcal{D}_k(P, Q)/\mathcal{D}_{k-1}(P, Q)$, is called the module of *symbols*. I denote by

$$\sigma_k : \mathcal{D}_k(P, Q) \rightarrow \mathcal{S}_k(P, Q)$$

the projection.

Let R be another A -module. The composition $\square_1 \circ \square_2 : P \rightarrow R$ of differential operators $\square_1 : Q \rightarrow R$ and $\square_2 : P \rightarrow Q$, of the order ℓ_1 and ℓ_2 , respectively, is a differential operator of the order $\ell_1 + \ell_2$. Accordingly, there is a well defined A -bilinear map

$$\odot : \mathcal{S}(Q, R) \times \mathcal{S}(P, Q) \rightarrow \mathcal{S}(P, R)$$

defined by

$$\sigma_{\ell_1}(\square_1) \odot \sigma_{\ell_2}(\square_2) := \sigma_{\ell_1 + \ell_2}(\square_1 \circ \square_2), \quad \square_i \in \mathcal{D}_{\ell_i}(P, Q), \quad i = 1, 2.$$

I simply denote by $\mathcal{D}(A) = \bigcup_k \mathcal{D}_k(A)$ (or just $\mathcal{D} = \bigcup_k \mathcal{D}_k$, if this does not lead to confusion) the graded, associative, filtered, unital K -algebra $\mathcal{D}(A, A)$ of differential operators $A \rightarrow A$ and by $\mathcal{S}(A)$ (or just \mathcal{S}) the corresponding module of symbols. The bilinear map $\mathcal{S}(A) \times \mathcal{S}(A) \rightarrow \mathcal{S}(A)$ defined above, gives $\mathcal{S}(A)$ the structure of a graded, associative, graded commutative, unital K -algebra. Notice that the (graded) commutator $[\square_1, \square_2]$ of differential operators $\square_1, \square_2 : A \rightarrow A$ of the order ℓ_1, ℓ_2 respectively, is a differential operator of the order $\ell_1 + \ell_2 - 1$. Accordingly, there is a well defined K -bilinear bracket

$$\{\cdot, \cdot\} : \mathcal{S}(A) \times \mathcal{S}(A) \rightarrow \mathcal{S}(A)$$

defined by

$$\{\sigma_{\ell_1}(\square_1), \sigma_{\ell_2}(\square_2)\} := \sigma_{\ell_1 + \ell_2 - 1}([\square_1, \square_2]), \quad \square_i \in \mathcal{D}_{\ell_i}, \quad i = 1, 2.$$

The bracket $\{\cdot, \cdot\}$ gives \mathcal{S} the structure of a graded, Poisson K -algebra. Notice that $\mathcal{D}_0 = \mathcal{S}_0 = A$, $\mathcal{D}_1 = A \oplus \text{Der} A$ and $\mathcal{S}_1 = \text{Der} A$, where $\text{Der} A$ denotes the A -module of derivations of A .

Denote by $\text{Der}_k(A, Q)$, the A -module of graded, graded symmetric, Q -valued multiderivations of A with k entries. The map

$$\varepsilon_k : \mathcal{S}_k(A, Q) \rightarrow \text{Der}_k(A, Q)$$

given by

$$\varepsilon_k \sigma_k(\square)(a_1, \dots, a_k) := \delta_{a_1} \cdots \delta_{a_k} \square, \quad \square \in \mathcal{D}_k, \quad a_1, \dots, a_k \in A$$

is a well defined A -linear map.

Remark 2. Let A be the \mathbb{R} -algebra of smooth functions on a graded manifold N . Then $\text{Der}_k(A, Q) \simeq Q \otimes S_A^k \text{Der} A$ and ε_k is an isomorphism of A -modules, whose inverse

$$Q \otimes S_A^k \text{Der} A \longrightarrow \mathcal{S}_k(A, Q)$$

is defined by

$$q \otimes X_1 \cdots X_k \longmapsto \sigma_k(q X_1 \circ \cdots \circ X_k),$$

$q \in Q, X_1, \dots, X_k \in \mathfrak{X}(N)$. Moreover, $(\mathcal{S}, \{\cdot, \cdot\})$ is the Poisson algebra of fiber-wise polynomial functions on T^*N .

1.2. Universal Enveloping of a Lie-Rinehart Algebra. Let $A = \bigoplus_i A_i$ be a graded, associative, graded commutative, unital K -algebra, and Q a graded *Lie-Rinehart algebra* over it, i.e., 1) Q is a graded Lie algebra and 2) an A -module, 3) A is a Q -module, and 4) the following compatibility conditions hold

$$\begin{aligned} (a \cdot q) \cdot b &= a \cdot (q \cdot b) \\ q \cdot (a \cdot b) &= (q \cdot a) \cdot b + (-)^{aq} a \cdot (q \cdot b) \\ [a \cdot q, r] &= a \cdot [q, r] - (-)^{r(a+q)} r(a) \cdot q \end{aligned}$$

for all $a, b \in A, q, r \in Q$. In particular Q acts on A via derivations. The prototype of a Lie-Rinehart algebra over A is $\text{Der} A$.

An *enveloping algebra* of the Lie-Rinehart algebra Q is a graded, associative, unital K -algebra E together with 1) a morphism $j : A \longrightarrow E$ of K -algebras, and 2) a morphism of Lie algebras $J : Q \longrightarrow E$ such that 3)

$$\begin{aligned} J(a \cdot q) &= j(a)J(q) \\ j(q \cdot a) &= J(q)j(a) - (-)^{aq} j(a)J(q) \end{aligned}$$

for all $a \in A, q \in Q$. As an example, notice that the associative algebra $\mathcal{D}(A)$ is an enveloping algebra of Q , with morphisms j, J given by the canonical injection $A \longrightarrow \mathcal{D}(A)$ and the action $Q \longrightarrow \text{Der} A \subset \mathcal{D}(A)$.

A *morphism of the enveloping algebras* E and E' is a morphism $f : E \longrightarrow E'$ of graded, unital K -algebras such that diagrams

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \nwarrow \quad \nearrow & \\ & A & \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \nwarrow \quad \nearrow & \\ & Q & \end{array}$$

commute. A *universal enveloping algebra* is an enveloping algebra $U(Q)$ such that for any other enveloping algebra E there is a unique morphism $U(Q) \longrightarrow E$ of enveloping algebras. In particular an enveloping algebra of Q acts on A by differential operators, i.e., there is a morphism of K -algebras

$$U(Q) \longrightarrow \mathcal{D}(A). \tag{1}$$

Universal enveloping algebras are clearly unique up to (unique) isomorphisms. A canonical one can be constructed as follows. Let \mathcal{U} be the tensor algebra of the graded vector space

$A \oplus Q$, and $I \subset \mathcal{U}$ the two sided ideal generated by relations

$$\begin{aligned} a \otimes b &= a \cdot b \\ a \otimes q &= a \cdot q \\ q \otimes a - (-)^{aq} a \otimes q &= q \cdot a \\ q \otimes r - (-)^{qr} r \otimes q &= [q, r], \end{aligned}$$

for all $a, b \in A$, and $q, r \in Q$. Put $U(Q) := \mathcal{U}/I$. Then $U(Q)$ is clearly a universal enveloping algebra of Q with morphisms j , and J given by the compositions of the canonical injections $A \rightarrow \mathcal{U}$, and $Q \rightarrow \mathcal{U}$, with the projection $\mathcal{U} \rightarrow U(Q)$.

It follows from the above construction that $U(Q)$ possesses an algebra filtration

$$U_0(Q) \subset U_1(Q) \subset \cdots \subset U_i(Q) \subset \cdots \subset U(Q) \quad (2)$$

bounded from below, where $U_i(Q) \subset U(Q)$ is the left A -submodule generated by products of at most i elements of the form $J(q)$, $q \in Q$. I denote by $\text{Gr}U(Q) = \bigoplus_i \text{Gr}_i U(Q)$ the graded algebra associated to the filtration (2), i.e., $\text{Gr}_i U(Q) := U_i(Q)/U_{i-1}(U)$. Since

$$[U_i(Q), U_j(Q)] \subset U_{i+j-1}(Q),$$

$\text{Gr}U(Q)$ is a commutative algebra, and the commutator in $U(Q)$ induce a graded Poisson bracket in it. Notice that $U_0(Q) = \text{Gr}_0 U(Q) = A$ and $U_1(Q) = \text{Gr}_1 U(Q) \oplus A$ where the splitting $\text{Gr}_1 U(Q) \rightarrow U_1(Q)$ of the exact sequence

$$0 \rightarrow A \rightarrow U_1(Q) \rightarrow \text{Gr}_1 U(Q) \rightarrow 0$$

is given by

$$\Delta + A \mapsto \Delta - \Delta(1).$$

There is a canonical A -linear, surjective, Poisson map

$$S_A^\bullet Q \rightarrow \text{Gr}U(Q) \quad (3)$$

mapping $S_A^i Q$ to $\text{Gr}_i U(Q)$, and given by

$$q_1 \odot \cdots \odot q_i \mapsto J(q_1) \cdots J(q_i) + U_{i-1}(Q).$$

Remark 3. If A is the algebra of smooth functions on a graded manifold N and Q is the Lie-Rinehart algebra of sections of a Lie algebroid over N then 1) projection (3) is an isomorphism, moreover 2) exact sequences $0 \rightarrow U_{i-1}(Q) \rightarrow U_i(Q) \rightarrow \text{Gr}_i U(Q) \rightarrow 0$ split (in a non canonical way). Therefore there is a (non-canonical) Poincaré-Birkhoff-Witt (PBW) type isomorphism of (filtered) A -modules

$$U(Q) \approx S_A^\bullet Q,$$

(for details about how to construct such isomorphism in the non-graded case see, for instance, [18]). Notice that, if $Q = \text{Der}A$ is the Lie-Rinehart algebra of vector fields over N , then (1) is an isomorphism and $U(Q)$ identifies with $\mathcal{D}(A)$ in a canonical way. Consequently, $\text{Gr}U(Q)$ identifies with the algebra $\mathcal{S}(A)$ of symbols.

Now, suppose that A is a commutative DG algebra with differential δ , and Q is a DG Lie-Rinehart algebra, i.e., Q is endowed with a degree 1 differential δ_0 such that 1) δ_0 is a

derivation of the graded Lie algebra structure, 2) δ_0 is a der-operator over δ of the A -module structure, i.e.,

$$\delta_0(a \cdot q) = \delta a \cdot q + (-)^a a \cdot \delta_0 q.$$

In the above hypothesis, δ and δ_0 can be extended as a unique derivation to the tensor algebra \mathcal{U} . Moreover, such derivation preserves the ideal I and, therefore, descend to a derivation of $U(Q)$ which becomes a DG algebra (satisfying a DG version of the universal properties of universal enveloping algebras) called the *universal enveloping DG algebra of the DG Lie-Rinehart algebra Q* .

Example 4. Let A be the DG algebra of smooth functions on a DG manifold N with homological vector field d , let $Q = \text{Der}A$, and let $\delta_0 : \text{Der}A \rightarrow \text{Der}A$ be the inner derivation $[d, \cdot]$. Then $U(Q)$ identifies with $\mathcal{D}(A)$ and the differential in it is again $[d, \cdot]$. Hence, $\text{Gr}U(Q)$ identifies with $S_A^\bullet Q$, the DG Poisson algebra of fiberwise polynomial functions on T^*N .

For more details about the material contained in this subsection see, for instance, [8, 17].

1.3. Strong Homotopy Structures. In this paper, conventions about strong homotopy algebras are the same as in [12]. Let (V, δ) be a complex of vector spaces and \mathcal{A} be any kind of algebraic structure (associative algebra, Lie algebra, module, etc.). Roughly speaking, a homotopy \mathcal{A} -structure in (V, δ) is an algebraic structure in V which is of the kind \mathcal{A} only up to δ -homotopies, and a *strong homotopy (SH) \mathcal{A} -structure* is a homotopy structure possessing a full system of (coherent) *higher homotopies*. In this paper, I will basically deal with four kinds of SH structures, namely SH associative algebras (also named A_∞ -algebras), SH modules (also named A_∞ -modules), SH Lie-Rinehart algebras, and SH Poisson algebras. For them I provide detailed definitions below.

Definition 5. Let \mathcal{A} be a graded vector space, and $\mathcal{A} = \{\alpha_k, k \in \mathbb{N}\}$ a family of k -ary, multilinear, degree $2 - k$ operations,

$$\alpha_k : \mathcal{A}^k \rightarrow \mathcal{A}, \quad k \in \mathbb{N}.$$

The pair $(\mathcal{A}, \mathcal{A})$ is an A_∞ -algebra if

$$\sum_{i+j=k} (-)^{ij} \sum_{\ell=0}^{i+j} (-)^{\ell(i+1)+i(x_1+\dots+x_\ell)} \alpha_{j+1}(x_1, \dots, x_\ell, \alpha_i(x_{\ell+1}, \dots, x_{\ell+i}), x_{\ell+i+1}, \dots, x_{i+j}) = 0$$

for all $x_1, \dots, x_k \in \mathcal{A}^k$, $k \in \mathbb{N}$ (in particular, (\mathcal{A}, α_1) is a cochain complex and $H(\mathcal{A}, \alpha_1)$ is a graded associative algebra).

If \mathcal{A} has only degree 0 homogeneous component, then an A_∞ -algebra structure in \mathcal{A} is simply an associative algebra structure for degree reasons. Similarly, if $\alpha_k = 0$ for all $k > 2$, then $(\mathcal{A}, \mathcal{A})$ is a DG (associative) algebra.

Definition 6. Let $(\mathcal{A}, \mathcal{A})$ be an A_∞ -algebra. A strict unit in \mathcal{A} is a degree 0 element $e \in \mathcal{A}$ such that $\alpha_2(e, x) = \alpha_2(x, e) = x$ for all $x \in \mathcal{A}$ and $\alpha_k = 0$, for all $k \neq 2$, whenever one of the entries is equal to e . An A_∞ -algebra with a strict unit is called *strictly unital*.

Definition 7. Let $(\mathcal{A}, \mathcal{A})$ be an A_∞ -algebra, $\mathcal{A} = \{\alpha_k, k \in \mathbb{N}\}$, M a graded vector space, and $\mathcal{M} = \{\mu_k, k \in \mathbb{N}\}$ a family of k -ary, multilinear, degree $2 - k$ operations,

$$\mu_k : \mathcal{A}^{k-1} \times M \rightarrow M, \quad k \in \mathbb{N}.$$

Define new operations

$$\alpha_k^\oplus : (\mathcal{A} \oplus M)^k \longrightarrow \mathcal{A} \oplus M, \quad k \in \mathbb{N},$$

extending the previous ones by linearity, and the condition that the result is zero if one of the first $k - 1$ entries is from M . The pair (M, \mathcal{M}) is an A_∞ -module over $(\mathcal{A}, \mathcal{A})$ if

$$\sum_{i+j=k} (-)^{ij} \sum_{\ell=0}^{i+j} (-)^{\ell(i+1)+i(y_1+\dots+y_\ell)} \alpha_{j+1}^\oplus(y_1, \dots, y_\ell, \alpha_i^\oplus(y_{\ell+1}, \dots, y_{\ell+i}), y_{\ell+i+1}, \dots, y_{i+j}) = 0$$

for all $y_1, \dots, y_k \in \mathcal{A} \oplus M$ (in particular, (M, μ_1) is a complex and $H(M, \mu_1)$ is a graded $H(\mathcal{A}, \alpha_1)$ -module).

If both \mathcal{A} and M have only degree 0 homogeneous component, then an A_∞ -module structure in M over \mathcal{A} is simply a left module structure over the associative algebra \mathcal{A} . Similarly, if $\alpha_k = 0$ and $\mu_k = 0$ for all $k > 2$, then (M, \mathcal{M}) is a DG module over the DG algebra \mathcal{A} .

Definition 8. Let L be a graded vector space, and $\mathcal{L} = \{\lambda_k, k \in \mathbb{N}\}$ a family of k -ary, graded skew-symmetric, multilinear, degree $2 - k$ operations,

$$\lambda_k : L^k \longrightarrow L, \quad k \in \mathbb{N}.$$

The pair (L, \mathcal{L}) is an L_∞ -algebra if

$$\sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_{i,j}} \chi(\sigma, \mathbf{v}) \lambda_{j+1}(\lambda_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(i+j)})_{j+1} = 0,$$

for all $\mathbf{v} = (v_1, \dots, v_k) \in L^k$, $k \in \mathbb{N}$ (in particular, (L, λ_1) is a complex and $H(L, \lambda_1)$ is a graded Lie algebra).

Notice that if L has only degree 0 homogeneous component, then an L_∞ -algebra structure in L is simply a Lie algebra structure. Similarly, if $\lambda_k = 0$ for all $k > 2$, then (L, \mathcal{L}) is a DG Lie algebra.

Definition 9. Let (L, \mathcal{L}) be an L_∞ -algebra, $\mathcal{L} = \{\lambda_k, k \in \mathbb{N}\}$, N a graded vector space, and $\mathcal{N} = \{\nu_k, k \in \mathbb{N}\}$ a family of k -ary, graded skew-symmetric (in the first $k - 1$ arguments), multilinear, degree $2 - k$ operations,

$$\nu_k : L^{k-1} \times N \longrightarrow N, \quad k \in \mathbb{N}.$$

Define new operations

$$\lambda_k^\oplus : (L \oplus N)^k \longrightarrow L \oplus N, \quad k \in \mathbb{N},$$

extending the previous ones by linearity, skew-symmetry, and the condition that the result is zero if more than one entry are from N . The pair (N, \mathcal{N}) is an L_∞ -module over (L, \mathcal{L}) if

$$\sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_{i,j}} \chi(\sigma, \mathbf{b}) \lambda_{j+1}^\oplus(\lambda_i^\oplus(b_{\sigma(1)}, \dots, b_{\sigma(i)}), b_{\sigma(i+1)}, \dots, b_{\sigma(i+j)})$$

for all $\mathbf{b} = (v_1, \dots, v_{k-1}, m) \in L^{k-1} \times N$, $k \in \mathbb{N}$ (in particular, (N, ν_1) is a complex and $H(N, \nu_1)$ is a graded $H(L, \lambda_1)$ -module).

If both L and N have only degree 0 homogeneous component, then an L_∞ -module structure in N over L is simply a Lie module structure over the Lie algebra L . Similarly, if $\lambda_k = 0$ and $\nu_k = 0$ for all $k > 2$, then (N, \mathcal{N}) is a DG Lie module over the DG Lie algebra L .

I now define SH Lie-Rinehart algebras [11]. For simplicity, I call the resulting objects LR_∞ -algebras.

Definition 10. Let (A, δ) be a commutative, unital DG algebra and let $(\mathcal{Q}, \mathcal{Q})$ be an L_∞ -algebra, $\mathcal{Q} = \{\lambda_k, k \in \mathbb{N}\}$. Furthermore, assume that \mathcal{Q} possesses the structure of an A -module, and A possesses the structure \mathcal{M} of an L_∞ -module over $(\mathcal{Q}, \mathcal{Q})$, $\mathcal{M} = \{\nu_k, k \in \mathbb{N}\}$, such that $\nu_1 = \delta$. The pair $(\mathcal{Q}, \mathcal{Q})$ is an LR_∞ -algebra over (A, δ) if

- (1) $\nu_{k+1} : \mathcal{Q}^k \times A \longrightarrow A$ is a derivation in the last argument, and A -multilinear in the first $k - 1$ arguments;
- (2) Formula

$$\lambda_k(q_1, \dots, q_{k-1}, aq_k) = \nu_k(q_1, \dots, q_{k-1} | a) \cdot q_k + (-)^{a(q_1 + \dots + q_{k-1} - k)} a \cdot \lambda_k(q_1, \dots, q_{k-1}, q_k), \quad (4)$$

holds for all $q_1, \dots, q_k \in \mathcal{Q}$, $a \in A$, $k \in \mathbb{N}$ (in particular, (\mathcal{Q}, λ_1) is a DG module over (A, δ) , and $H(\mathcal{Q}, \lambda_1)$ is a graded Lie-Rinehart algebra over $H(A, \delta)$).

If \mathcal{Q} and A have only degree 0 homogeneous component, then A is simply an associative, commutative algebra, and \mathcal{Q} a Lie-Rinehart algebra over it. Similarly, if $\lambda_k = 0$ for all $k > 2$, then $(\mathcal{Q}, \mathcal{Q})$ is a DG Lie-Rinehart algebra over (A, δ) .

In the *smooth setting*, i.e., when (A, δ) is the DG algebra of smooth functions on a DG manifold (see, for instance, [4]), \mathcal{Q} is sometimes called (the module of sections of) an L_∞ -algebroid [19, 3].

Definition 11. Let $(\mathcal{P}, \mathcal{P})$ be an L_∞ -algebra, $\mathcal{P} = \{\Lambda_k, k \in \mathbb{N}\}$. Furthermore, assume that \mathcal{P} possesses the structure of a graded, associative, graded commutative, unital algebra. The pair $(\mathcal{P}, \mathcal{P})$ is a SH Poisson algebra if Λ_k is a graded multiderivation for all $k \in \mathbb{N}$.

Notice that if \mathcal{P} has only degree 0 homogeneous component, then a SH Poisson algebra structure in \mathcal{P} is simply a Poisson algebra structure. Similarly, if $\lambda_k = 0$ for all $k > 2$, then $(\mathcal{P}, \mathcal{P})$ is a DG Poisson algebra.

Remark 12. Let (A, δ) be a commutative, unital DG algebra and \mathcal{Q} an A -module. The datum of an LR_∞ -algebra structure in \mathcal{Q} over (A, δ) is equivalent to the datum of a SH Poisson algebra structure in $S_A^\bullet \mathcal{Q}$, restricting to (A, δ) [3]. The operations in $S_A^\bullet \mathcal{Q}$ can be obtained from the ones in \mathcal{Q} , extending the latter as multiderivations.

Finally, notice that the canonical construction of a Lie algebra from an associative algebra can be generalized to the SH context as follows. Let $(\mathcal{A}, \mathcal{A})$ be an A_∞ -algebra, $\mathcal{A} = \{\alpha_k, k \in \mathbb{N}\}$. Define new operations

$$A\alpha_k : \mathcal{A}^k \longrightarrow \mathcal{A},$$

by putting

$$(A\alpha_k)(x_1, \dots, x_k) := \sum_{\sigma \in S_k} \chi(\sigma, \mathbf{x}) \alpha_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}),$$

$\mathbf{x} = (x_1, \dots, x_k) \in \mathcal{A}^k$, i.e., $A\alpha_k$ is the *skew-symmetrization* of α_k . The $A\alpha$'s give to \mathcal{A} the structure of an L_∞ -algebra [13].

Remark 13. *The theory of universal enveloping of L_∞ -algebras (see, for instance, [2]) is not fully developed, not to speak about universal enveloping of LR_∞ -algebras. However, few (naive) remarks can be done in this respect. First of all, recall that a morphism $f : \mathcal{A} \rightarrow \mathcal{A}'$ (resp., $f : L \rightarrow L'$) of A_∞ -algebras (resp., L_∞ -algebras) is a family of K -multilinear (resp., skew-symmetric, K -multilinear) maps $f_k : \mathcal{A}^k \rightarrow \mathcal{A}'$ (resp., $f_k : L^k \rightarrow L'$) satisfying suitable compatibility conditions (see, for instance, [12] for details). It is tempting to define an enveloping SH algebra for an LR_∞ -algebra \mathcal{Q} over a DG algebra A , as an A_∞ -algebra \mathcal{E} together with 1) a morphism of DG algebras $j : A \rightarrow \mathcal{E}$, and 2) a morphism of L_∞ -algebras $J : \mathcal{Q} \rightarrow \mathcal{E}$ such that 3)*

$$J_k(a \cdot q_1, q_2, \dots, q_k) = j(a)J_k(q_1, \dots, q_k)$$

and

$$\begin{aligned} & j(\nu_k(q_1, \dots, q_{k-1}|a)) \\ &= \sum_{\ell=1}^{k-1} \sum_{\substack{k_1+\dots+k_\ell=k-1 \\ k_1 \leq \dots \leq k_\ell}} \sum_{\sigma \in S_{k_1, \dots, k_\ell}^<} \chi(\sigma, \mathbf{q})(A\alpha_{\ell+1})(J_{k_1}(q_{\sigma(1)}, \dots), \dots, J_{k_\ell}(\dots, q_{\sigma(k-1)}), ja), \end{aligned} \quad (5)$$

(here $S_{k_1, \dots, k_\ell}^< \subset S_{k_1, \dots, k_\ell}$ is the set of (k_1, \dots, k_ℓ) -unshuffles such that

$$\sigma(k_1 + \dots + k_{i-1} + 1) < \sigma(k_1 + \dots + k_{i-1} + k_i + 1) \quad \text{whenever } k_i = k_{i+1},$$

see the definition of morphism of L_∞ -algebras, e.g., in [12]). One could then define a universal enveloping SH algebra as an enveloping SH algebra satisfying (obvious) universal properties, and try to construct it. Developing this ideas goes beyond the scopes of this paper.

1.4. Homological Perturbations and Homotopy Transfer. The main homological tools used in this paper are the *Perturbation Lemma* and the *Homotopy Transfer Theorem*. I recall in this section those versions of them that will be used below.

Let (\mathcal{C}, δ_0) and $(\underline{\mathcal{C}}, \underline{\delta})$ be cochain complexes of vector spaces, $p : (\mathcal{C}, \delta) \rightarrow (\underline{\mathcal{C}}, \underline{\delta})$ and $j : (\underline{\mathcal{C}}, \underline{\delta}) \rightarrow (\mathcal{C}, \delta_0)$ cochain maps and $h : \mathcal{C} \rightarrow \mathcal{C}$ a degree -1 endomorphism:

$$h \circ \begin{matrix} \mathcal{C} \\ \delta \end{matrix} \xrightleftharpoons[j]{p} \begin{matrix} \underline{\mathcal{C}} \\ \underline{\delta} \end{matrix}$$

Definition 14. *The data (p, j, h) are contraction data for (\mathcal{C}, δ_0) over $(\underline{\mathcal{C}}, \underline{\delta})$ if*

- (1) *j is a right inverse of p , i.e., $pj = \text{id}$,*
- (2) *h is a contracting homotopy, i.e., $[h, \delta] = \text{id} - jp$,*
- (3) *the side conditions $h^2 = 0$, $hj = 0$, $ph = 0$ are satisfied.*

Now, let (p_0, j_0, h_0) be contraction data for a cochain complex (\mathcal{C}, δ_0) over $(\underline{\mathcal{C}}, \underline{\delta})$. Suppose that there is another differential δ in \mathcal{C} , and put $t := \delta_0 - \delta$. The Perturbation Lemma allows one to construct contraction data for (\mathcal{C}, δ) over a suitable new complex $(\underline{\mathcal{C}}, \underline{\delta}_t)$.

Theorem 15 (Perturbation Lemma). *Let $th_0 : \mathcal{C} \rightarrow \mathcal{C}$ be locally nilpotent, i.e., for any $x \in \mathcal{C}$ there is $k \in \mathbb{N}$ such that $(th_0)^k(x) = 0$, and*

$$X := t + th_0t + th_0th_0t + \dots = \sum_{i=0}^{\infty} t(h_0t)^i = \sum_{i=0}^{\infty} (th_0)^i t.$$

Moreover, let $\underline{\delta}_t, p_t, j_t, h_t$ be defined as

$$\begin{aligned}\underline{\delta}_t &:= \underline{\delta} - p_0 X j_0 \\ p_t &:= p_0(\text{id} + X h_0)\end{aligned}\tag{6}$$

$$j_t := (\text{id} + h_0 X) j_0 \tag{7}$$

$$h_t := h_0 + h_0 X h_0. \tag{8}$$

Then (p_t, j_t, h_t) are contraction data for (\mathcal{C}, δ) over $(\underline{\mathcal{C}}, \underline{\delta}_t)$.

Remark 16. A rather standard situation, which will be also encountered in this paper, is when \mathcal{C} and $\underline{\mathcal{C}}$ are endowed with filtrations

$$\begin{aligned}\mathcal{C}_0 &\subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_i \subset \cdots \subset \mathcal{C}, \\ \underline{\mathcal{C}}_0 &\subset \underline{\mathcal{C}}_1 \subset \cdots \subset \underline{\mathcal{C}}_i \subset \cdots \subset \underline{\mathcal{C}},\end{aligned}$$

bounded from below, and such that, 1) they are preserved by $\delta_0, \underline{\delta}, p_0, j_0, h_0$, and 2) $t(\mathcal{C}_i) \subset \mathcal{C}_{i-1}$. In this case, h_0 is automatically locally nilpotent and the Perturbation Lemma applies.

Contraction data for (\mathcal{C}, δ) over $(\underline{\mathcal{C}}, \underline{\delta})$ can be used to transfer SH structures from the former to the latter, in particular when the SH structure one begins with does not possess higher homotopies. This is a rather rich source of SH structures. Moreover, there are explicit formulas for the higher homotopies of the induced structure.

Theorem 17 (Homotopy Transfer Theorem, see, e.g., [15]). *Let (\mathcal{C}, δ) and $(\underline{\mathcal{C}}, \underline{\delta})$ be cochain complexes and let (p, j, h) be contraction data for (\mathcal{C}, δ) over $(\underline{\mathcal{C}}, \underline{\delta})$.*

- (1) *Assume (\mathcal{C}, δ) possesses the structure \circ of a DG algebra, and let $\mathcal{A} = \{\alpha_k, k \in \mathbb{N}\}$ be the family of graded operations*

$$\alpha_k : \underline{\mathcal{C}}^k \longrightarrow \underline{\mathcal{C}}$$

defined by

$$\alpha_1 := \underline{\delta}, \quad \alpha_k := p\beta_k, \quad k \geq 2,$$

where the β 's are inductively defined by

$$\gamma_1 := -j, \quad \gamma_k := h\beta_k,$$

and

$$\beta_k(x_1, \dots, x_k) := \sum_{\ell+m=k} (-)^{a(\ell, m, \mathbf{x})} \gamma_\ell(x_1, \dots, x_\ell) \circ \gamma_m(x_{\ell+1}, \dots, x_{\ell+m}),$$

$x_1, \dots, x_k \in \underline{\mathcal{C}}$, where $a(\ell, m, \mathbf{x}) := \ell - 1 + (m - 1) \sum_{i=1}^\ell \bar{x}_i$, $k \geq 2$. Then $(\underline{\mathcal{C}}, \mathcal{A})$ is an A_∞ -algebra. Moreover, if (\mathcal{C}, δ) is a unital DG algebra with unit $1_{\mathcal{C}}$ such that $(jp)1_{\mathcal{C}} = 1_{\mathcal{C}}$, then $(\underline{\mathcal{C}}, \mathcal{A})$ is a strictly unital A_∞ -algebra with unit $p1_{\mathcal{C}}$.

- (2) *Assume (\mathcal{C}, δ) possesses the structure $[\cdot, \cdot]$ of a DG Lie algebra, and let $\mathcal{L} = \{\lambda_k, k \in \mathbb{N}\}$ be the family of graded operations*

$$\lambda_k : \underline{\mathcal{C}}^k \longrightarrow \underline{\mathcal{C}}$$

defined by

$$\lambda_1 := \underline{\delta}, \quad \lambda_k := p\phi_k, \quad k \geq 2 \tag{9}$$

where the ϕ 's are inductively defined by

$$\psi_1 := -j, \quad \psi_k := h\phi_k,$$

and

$$\begin{aligned} & \phi_k(x_1, \dots, x_k) \\ &:= \sum_{\ell+m=k} \sum_{\sigma \in S_{\ell, m}} (-)^{b(\ell, m, \mathbf{x})} \chi(\sigma, \mathbf{x}) [\psi_\ell(x_{\sigma(1)}, \dots, x_{\sigma(\ell)}), \psi_m(x_{\sigma(\ell+1)}, \dots, x_{\sigma(\ell+m)})], \end{aligned}$$

$x_1, \dots, x_k \in \underline{\mathcal{C}}$, where $b(\ell, m, \mathbf{x}) := \ell - 1 + (m - 1) \sum_{\sigma(i)=1}^{\ell} x_{\sigma(i)}$, $k \geq 2$. Then $(\underline{\mathcal{C}}, \mathcal{L})$ is an L_∞ -algebra.

1.5. Homotopy Transfer of Universal Enveloping. In this subsection I present an abstract algebraic model for the concrete geometric framework of the next section.

SH module structures can be transferred along contraction data similarly as in the previous subsection. Even more, one can transfer a SH Lie-Rinehart algebra structure along suitable contraction data. Namely, let (A, δ) be a commutative, unital DG algebra, let \mathcal{C} be a DG Lie-Rinehart algebra over (A, δ) with differential δ_0 and Lie-bracket $[\cdot, \cdot]$, and let $\underline{\mathcal{C}}$ be a DG-module over (A, δ) with differential $\underline{\delta}$. Moreover, suppose that there are A -linear contraction data (p_0, j_0, h_0) for (\mathcal{C}, δ_0) over $(\underline{\mathcal{C}}, \underline{\delta})$. Then, it is easy to see that there is an LR_∞ -algebra structure \mathcal{Q} in $\underline{\mathcal{C}}$ defined in a similar way as in Theorem 17. I do not report here the obvious details.

Now, consider the symmetric DG algebras $S_A^\bullet \mathcal{C}$ and $S_A^\bullet \underline{\mathcal{C}}$. In view of Remark 12, they are endowed with a DG Poisson structure and a SH Poisson algebra structure \mathcal{P} , respectively. I denote 1) by $\{\cdot, \cdot\}$ the Poisson bracket in $S_A^\bullet \mathcal{C}$, and 2) again by δ_0 and $\underline{\delta}$ the differentials in $S_A^\bullet \mathcal{C}$ and $S_A^\bullet \underline{\mathcal{C}}$, respectively. I claim that the contraction data (p_0, j_0, h_0) extend to contraction data

$$h_0 \circlearrowleft (S_A^\bullet \mathcal{C}, \delta_0) \xrightleftharpoons[j_0]{p_0} (S_A^\bullet \underline{\mathcal{C}}, \underline{\delta})$$

such that the above mentioned SH Poisson algebra structure in $S_A^\bullet \underline{\mathcal{C}}$ is obtained from the DG Poisson structure in $S_A^\bullet \mathcal{C}$ via homotopy transfer. Indeed, put $\mathcal{K} := \ker p_0$. Then $\mathcal{C} = \underline{\mathcal{C}} \oplus \mathcal{K}$ and $S_A^\bullet \mathcal{C} \simeq S_A^\bullet \underline{\mathcal{C}} \otimes S_A^\bullet \mathcal{K}$. Now, extend p_0 and j_0 as algebra morphisms, and let $h' : S_A^\bullet \mathcal{C} \rightarrow S_A^\bullet \mathcal{C}$ be the extension of h_0 as a derivation. Now, for $\Sigma \in S_A^\bullet \underline{\mathcal{C}} \otimes S_A^i \mathcal{K} \subset S_A^\bullet \mathcal{C}$, put

$$h_0 \Sigma := \begin{cases} 0 & \text{if } i = 0 \\ \frac{1}{i} h' \Sigma & \text{if } i > 0 \end{cases}.$$

It is easy to see that (p_0, j_0, h_0) are contraction data for $(S_A^\bullet \mathcal{C}, \delta_0)$ over $(S_A^\bullet \underline{\mathcal{C}}, \underline{\delta})$ extending the previous ones. Thus, in view of the Homotopy Transfer Theorem, there is an L_∞ -algebra structure $\mathcal{L} = \{\lambda_k, k \in \mathbb{N}\}$ in $S_A^\bullet \underline{\mathcal{C}}$ given by Formulas (9). Notice that $h_0 : S_A^\bullet \mathcal{C} \rightarrow S_A^\bullet \mathcal{C}$ is $S_A^\bullet \underline{\mathcal{C}}$ -linear, i.e.,

$$h_0(j_0 \Sigma \odot \Sigma') = (-)^\Sigma j_0 \Sigma \odot h_0 \Sigma', \quad \text{for all } \Sigma \in S_A^\bullet \underline{\mathcal{C}}, \Sigma' \in S_A^\bullet \mathcal{C}.$$

Proposition 18. *The structures \mathcal{P} and \mathcal{L} coincide.*

Proof. Since \mathcal{L} extends \mathcal{Q} , it is enough to show that the λ 's are multiderivations. This can be proved by induction as follows. I claim that, for any k , ϕ_k is an “approximate” multiderivation along j_0 in the following sense:

$$\begin{aligned} & \phi_k(\Sigma' \odot \Sigma'', \Sigma_1, \dots, \Sigma_{k-1}) \\ &= (-)^{\Sigma'k} j_0 \Sigma' \odot \phi_k(\Sigma'', \Sigma_1, \dots, \Sigma_{k-1}) + (-)^{\Sigma''(\Sigma_1 + \dots + \Sigma_{k-1})} \phi_k(\Sigma', \Sigma_1, \dots, \Sigma_{k-1}) \odot j_0 \Sigma'' + I \end{aligned} \quad (10)$$

for all $\Sigma', \Sigma'', \Sigma_1, \dots, \Sigma_{k-1} \in S_A^\bullet \mathcal{C}$, where I is the ideal of $(S_A^\bullet \mathcal{C}, \odot)$ generated by the image of h_0 . Since $I \subset \ker p_0$, it follows from my claim and the side condition $p_0 h_0 = 0$ that λ_k is a multiderivation. Now, prove the claim by induction on k . First of all, a straightforward computation shows that

$$\phi_2(\Sigma' \odot \Sigma'', \Sigma) = j_0 \Sigma' \odot \phi_2(\Sigma'', \Sigma) + (-)^{\Sigma \Sigma''} \phi_2(\Sigma', \Sigma) \odot j_0 \Sigma''.$$

Now, assume that (2) holds for all $k \leq n$, and prove it for $k = n + 1$. From skew-symmetry it is enough to check it on equal, odd elements $\Sigma_1 = \dots = \Sigma_n = \Sigma$. Put $\Sigma := (\Sigma' \odot \Sigma'', \Sigma^n)$ and compute

$$\begin{aligned} & \phi_{n+1}(\Sigma' \odot \Sigma'', \Sigma^n) \\ &= 2 \sum_{\ell+m=n} (-)^{b(\ell, m, \Sigma)} \binom{\ell+m}{\ell} \{ \psi_{\ell+1}(\Sigma' \odot \Sigma'', \Sigma^\ell), \psi_m(\Sigma^m) \} \\ &= -2 \{ j_0 \Sigma' \odot j_0 \Sigma'', h_0 \phi_n(\Sigma^n) \} \\ &\quad + 2 \sum_{\ell=1}^{n-1} (-)^{b(\ell, n-\ell, \Sigma)} \binom{n}{\ell} \{ h_0 \phi_{\ell+1}(\Sigma' \odot \Sigma'', \Sigma^\ell), h_0 \phi_{n-\ell}(\Sigma^{n-\ell}) \} + I \\ &= -2 j_0 \Sigma' \odot \{ j_0 \Sigma'', h_0 \phi_n(\Sigma^n) \} - 2 (-)^{\Sigma''} \{ j_0 \Sigma'', h_0 \phi_n(\Sigma^n) \} \odot j_0 \Sigma'' \\ &\quad + 2 \sum_{\ell=1}^{n-1} (-)^{b(\ell, n-\ell, \Sigma)} \binom{n}{\ell} [(-)^{\ell \Sigma'} \{ j_0 \Sigma' \odot h_0 \phi_{\ell+1}(\Sigma'', \Sigma^\ell), h_0 \phi_{n-\ell}(\Sigma^{n-\ell}) \} \\ &\quad + (-)^{\ell \Sigma''} \{ h_0 \phi_{\ell+1}(\Sigma', \Sigma^\ell) \odot j_0 \Sigma'', h_0 \phi_{n-\ell}(\Sigma^{n-\ell}) \}] + I \\ &= (-)^{\Sigma'(n+1)} j_0 \Sigma' \odot \phi_{n+1}(\Sigma'', \Sigma^n) + (-)^{n \Sigma''} \phi_{n+1}(\Sigma', \Sigma^n) \odot j_0 \Sigma'' + I \end{aligned}$$

where I used the fact that, since $h_0(I) \subset I \odot I$, then $\{h_0(I), S_A^\bullet \mathcal{C}\} \subset I$. \square

Now, let $(U(\mathcal{C}), \delta)$ be the universal enveloping DG algebra of (\mathcal{C}, δ_0) , and suppose there is a PBW type isomorphism $U(\mathcal{C}) \approx S_A^\bullet \mathcal{C}$, i.e., an isomorphism $\text{PBW} : S_A^\bullet \mathcal{C} \rightarrow U(\mathcal{C})$ of filtered A -modules such that diagram

$$\begin{array}{ccc} S_A^{\leq k} \mathcal{C} & \xrightarrow{\text{PBW}} & U_k(\mathcal{C}) \\ & \searrow & \swarrow \\ & \text{Gr}_k U(\mathcal{C}) & \end{array}$$

commutes for all k , (the map $S_A^{\leq k} \mathcal{C} \rightarrow \text{Gr}_k U(\mathcal{C})$ being given by the composition of projections $S_A^{\leq k} \mathcal{C} \rightarrow S_A^k \mathcal{C}$ and $S_A^k \mathcal{C} \rightarrow \text{Gr}_k U(\mathcal{C})$). Use PBW to identify $U(\mathcal{C})$ and $S_A^\bullet \mathcal{C}$. Then 1) the filtrations in $U(\mathcal{C}) = S_A^\bullet \mathcal{C}$ and $S_A^\bullet \mathcal{C}$ are preserved by $\delta_0, \underline{\delta}, p_0, j_0, h_0$, and 2) $t(U_i(\mathcal{C})) \subset U_{i-1}(\mathcal{C})$. It follows from Remark 16 and the Perturbation Lemma that there are contraction data (p_t, j_t, h_t) for $(U(\mathcal{C}), \delta)$ over $(S_A^\bullet \mathcal{C}, \underline{\delta}_t)$. Hence, in view of the Homotopy Transfer Theorem,

there is an A_∞ -algebra structure in $S_A^\bullet \underline{\mathcal{C}}$ canonically determined by the contraction data (p_0, j_0, h_0) and PBW.

Example 19. Let M be a smooth manifold, \mathcal{F} a foliation of M , and C its characteristic distribution. Moreover, let (\mathcal{C}, δ_0) be the deformation complex of \mathcal{F} and $(\underline{\mathcal{C}}, \underline{\delta})$ the Chevalley-Eilenberg complex determined by the Bott connection in TM/C . A splitting $TM = C \oplus V$ via a complementary distribution V determines contraction data (p_0, j_0, h_0) for (\mathcal{C}, δ_0) over $(\underline{\mathcal{C}}, \underline{\delta})$. Accordingly, there is an LR_∞ -algebra structure in $\underline{\mathcal{C}}$ which I described in [24] (see also [9, 10]). In Subsection 2.4 I show how to construct a PBW isomorphism $U(\mathcal{C}) \approx S^\bullet \mathcal{C}$, via purely geometric data (specifically, a connection). One immediately concludes that there is an A_∞ -algebra structure in $S^\bullet \underline{\mathcal{C}}$. I partially describe this A_∞ -algebra in Section 3. Here, I present the toy example when \mathcal{F} has just one leaf and $C = TM$, as an illustration of the main technical aspects of the general case.

When $C = TM$, the deformation complex of \mathcal{F} is $(\text{Der}\Lambda(M), \delta_0 = [d, \cdot])$, $TM/C = 0$, and its Chevalley-Eilenberg complex $(\underline{\mathcal{C}}, \underline{\delta})$ is trivial. Put $\Lambda := \Lambda(M)$. There are contraction data $(0, 0, h_0)$ for $(\text{Der}\Lambda, \delta_0)$ over the 0 complex. The contracting homotopy h_0 is defined as follows. Every element $\Delta \in \text{Der}\Lambda$ can be uniquely written as [16] $\Delta = i_U + L_V$, $U, V \in \Lambda \otimes \mathfrak{X}(M)$. Then $h_0(\Delta) := (-)^\Delta i_V$. The homotopy h_0 is Λ -linear. Accordingly, it determines contraction data (p_0, j_0, h_0) for $(\mathcal{S}(\Lambda) = S_A^\bullet \text{Der}\Lambda, \delta_0)$ over $(S_A^\bullet \underline{\mathcal{C}}, \underline{\delta}) = (\Lambda, d)$, where $p_0 : S_A^\bullet \text{Der}\Lambda \rightarrow \Lambda$ and $j_0 : \Lambda \rightarrow S_A^\bullet \text{Der}\Lambda$ are the obvious maps. Notice that the SH Poisson algebra structure induced on (Λ, d) is trivial. The universal enveloping DG algebra of $\text{Der}\Lambda$ is $(\mathcal{D}(\Lambda), \delta = [d, \cdot])$. A PBW isomorphism $\mathcal{D}(\Lambda) \approx \mathcal{S}(\Lambda)$ can be constructed, exploiting a connection ∇ , as follows (see [7, 18] for similar results). Extend the covariant derivative $\nabla : \mathfrak{X}(M) \rightarrow \text{Der}\Lambda$ to the whole $\Lambda \otimes \mathfrak{X}(M)$ by Λ -linearity. For $Z \in \Lambda \otimes \mathfrak{X}(M)$, $L_Z - \nabla_Z = i_{\nabla Z}$. It follows that every element Δ in $\text{Der}\Lambda$ can be uniquely written in the form $\Delta = i_U + \nabla_Z$, $U, Z \in \Lambda \otimes \mathfrak{X}(M)$, and the correspondence

$$\Lambda \otimes \mathfrak{X}(M)[1] \oplus \Lambda \otimes \mathfrak{X}(M) \ni (U, Z) \longmapsto i_U + \nabla_Z \in \text{Der}\Lambda$$

is a well defined isomorphism of Λ -modules. Accordingly, $\mathcal{S}(\Lambda)$ identifies with

$$S_A^\bullet(\Lambda \otimes \mathfrak{X}(M)[1]) \otimes_{\Lambda} S_A^\bullet(\Lambda \otimes \mathfrak{X}(M)) \simeq \Lambda \otimes \Lambda^\bullet \mathfrak{X}(M) \otimes S^\bullet \mathfrak{X}(M).$$

Now, let

$$\Sigma = \omega \otimes Y_1 \wedge \cdots \wedge Y_j \otimes P \in \Lambda \otimes \Lambda^j \mathfrak{X}(M) \otimes S^\ell \mathfrak{X}(M),$$

let \dots, z^a, \dots be coordinates in M , and let P be locally given by $P = P^{a_1 \cdots a_\ell} \frac{\partial}{\partial z^{a_1}} \odot \cdots \odot \frac{\partial}{\partial z^{a_\ell}}$. Define $\nabla_P : \Lambda \rightarrow \Lambda$ via local formulas $\nabla_P := P^{a_1 \cdots a_\ell} \nabla_{a_1} \cdots \nabla_{a_\ell}$, and put

$$\text{PBW}(\Sigma) := \omega i_{Y_1} \cdots i_{Y_j} \nabla_P \in \mathcal{D}_{j+\ell}(\Lambda). \quad (11)$$

The restrictions $\text{PBW} : \mathcal{S}_i(\Lambda) \rightarrow \mathcal{D}_i(\Lambda)$ split the exact sequences $0 \rightarrow \mathcal{D}_{i-1}(\Lambda) \rightarrow \mathcal{D}_i(\Lambda) \rightarrow \mathcal{S}_i(\Lambda) \rightarrow 0$, so that PBW is the required PBW isomorphism. The Perturbation Lemma gives now contraction data for $(\mathcal{D}(\Lambda), \delta)$ over (Λ, d) . The A_∞ -algebra structure induced on (Λ, d) is again trivial.

2. GEOMETRIC PRELIMINARIES

2.1. (A Bit of) Differential Geometry and Homological Algebra of a Foliation. Let M be a smooth manifold and C an involutive n -dimensional distribution on it. Now on, I will

denote by A the algebra of smooth functions on M . I will denote by $C\mathfrak{X}$ the submodule of $\mathfrak{X}(M)$ made of vector fields in C . Let $C\Lambda^1 := C\mathfrak{X}^\perp \subset \Lambda^1(M)$ be its annihilator, and put

$$\overline{\mathfrak{X}} := \mathfrak{X}(M)/C\mathfrak{X}, \quad \overline{\Lambda}^1 := \Lambda^1(M)/C\Lambda^1.$$

Then $C\Lambda^1 \simeq \overline{\mathfrak{X}}^*$ and $\overline{\Lambda}^1 \simeq C\mathfrak{X}^*$. In view of the Fröbenius theorem, there always exist coordinates $\dots, x^i, \dots, u^\alpha, \dots$, $i = 1, \dots, n$, $\alpha = 1, \dots, \dim M - n$, adapted to C , i.e., such that $C\mathfrak{X}$ is locally spanned by $\dots, \partial_i := \partial/\partial x^i, \dots$ and $C\Lambda^1$ is locally spanned by \dots, du^α, \dots . Consider the Chevalley-Eilenberg algebra $(\overline{\Lambda}, \overline{d})$ of the Lie algebroid $C\mathfrak{X}$. Namely, $\overline{\Lambda}$ is the exterior algebra of $\overline{\Lambda}^1$ and

$$(\overline{d}\lambda)(X_1, \dots, X_{k+1}) = \sum_i (-)^{i+1} X_i(\lambda(\dots, \widehat{X}_i, \dots)) + \sum_{i < j} (-)^{i+j} \lambda([X_i, X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots),$$

where $\lambda \in \overline{\Lambda}^k$ is understood as a $C^\infty(M)$ -valued, k -multilinear, skew-symmetric map on $C\mathfrak{X}$ and $X_1, \dots, X_{k+1} \in C\mathfrak{X}$. The DG algebra $(\overline{\Lambda}, \overline{d})$ is the quotient of $(\Lambda(M), d)$ over the differentially closed ideal generated by $C\Lambda^1$ which is made of differential forms vanishing when acting on vector fields in $C\mathfrak{X}$. In particular, it is generated by degree 0, and \overline{d} -exact degree 1 elements.

In the following, I denote by

$$\Lambda(M) \ni \omega \longmapsto \overline{\omega} \in \overline{\Lambda} \tag{12}$$

the projection.

The Lie algebroid $C\mathfrak{X}$ acts on $\overline{\mathfrak{X}}$ via the *Bott connection*. Namely, denote by

$$\mathfrak{X}(M) \ni X \longmapsto \overline{X} \in \overline{\mathfrak{X}} \tag{13}$$

the projection. Then

$$X \cdot \overline{Y} := \overline{[X, Y]} \in \overline{\mathfrak{X}}, \quad X \in C\mathfrak{X}, Y \in \mathfrak{X}(M).$$

Accordingly, there is a DG module $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \overline{d})$ over $(\overline{\Lambda}, \overline{d})$ whose differential is given by the usual Chevalley-Eilenberg formula:

$$\begin{aligned} (\overline{d}Z)(X_1, \dots, X_{k+1}) \\ = \sum_i (-)^{i+1} X_i \cdot Z(\dots, \widehat{X}_i, \dots) + \sum_{i < j} (-)^{i+j} Z([X_i, X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots), \end{aligned}$$

where $Z \in \overline{\Lambda}^k \otimes \overline{\mathfrak{X}}$ is understood as a $\overline{\mathfrak{X}}$ -valued, k -multilinear, skew-symmetric map on $C\mathfrak{X}$, and $X_1, \dots, X_{k+1} \in C\mathfrak{X}$. The tensor product of projections (12) and (13) is similarly denoted by

$$\Lambda(M) \otimes \mathfrak{X}(M) \ni Z \longmapsto \overline{Z} \in \overline{\Lambda} \otimes \overline{\mathfrak{X}}.$$

Remark 20. The differentials \overline{d} in $\overline{\Lambda}$ and $\overline{\mathfrak{X}}$ can be uniquely extended to the whole tensor algebra

$$\bigoplus_{i,j} \overline{\Lambda} \otimes \overline{\mathfrak{X}}^{\otimes i} \otimes (C\Lambda^1)^{\otimes j},$$

requiring Leibniz rules with respect to tensor products and contractions. Such extension is nothing but the Chevalley-Eilenberg differential associated to the canonical action of $C\mathfrak{X}$ on $\bigoplus_{i,j} \overline{\mathfrak{X}}^{\otimes i} \otimes (C\Lambda^1)^{\otimes j}$. In particular, \overline{d} extends to an homological derivation \overline{d}_S of $\overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$.

The exact sequence

$$0 \longrightarrow C\mathfrak{X} \longrightarrow \mathfrak{X}(M) \longrightarrow \overline{\mathfrak{X}} \longrightarrow 0$$

splits. The datum of a splitting is equivalent to the datum of a distribution V complementary to C . From now on fix such a distribution. I will always identify $\overline{\mathfrak{X}}$ (resp., $\overline{\Lambda}$) with the corresponding submodule (resp., subalgebra) in $\mathfrak{X}(M)$ (resp., $\Lambda(M)$) determined by V .

The distribution $V \simeq TM/C$ is locally spanned by vector fields \dots, V_α, \dots of the form $V_\alpha := \partial/\partial u^\alpha + V_\alpha^i \partial_i$, $\alpha = 1, \dots, \dim M - n$, for some local functions \dots, V_α^i, \dots . Moreover,

$$[\partial_i, V_\alpha] = \partial_i V_\alpha^j \partial_j, \quad [V_\alpha, V_\beta] = R_{\alpha\beta}^i \partial_i, \quad (14)$$

where $R_{\alpha\beta}^i := V_\alpha V_\beta^i - V_\beta V_\alpha^i$.

Now, consider the *deformation complex* $(\text{Der}\overline{\Lambda}, \delta_0 := [\overline{d}, \cdot])$ (see, for instance, [6]) of the integral foliation of C . The complementary distribution V determines contraction data (p_0, j_0, h_0) for $(\text{Der}\overline{\Lambda}, \delta_0)$ over $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \overline{d})$. Accordingly, there is an LR_∞ -algebra structure in $\overline{\Lambda} \otimes \overline{\mathfrak{X}}$ (see the second appendix of [24]). Recall that the projection $p_0 : \text{Der}\overline{\Lambda} \longrightarrow \overline{\Lambda} \otimes \overline{\mathfrak{X}}$ is actually independent of V and is defined as

$$p_0 \Delta := \overline{\Delta|_A}, \quad \Delta \in \text{Der}\overline{\Lambda}. \quad (15)$$

The injection $j_0 : \overline{\Lambda} \otimes \overline{\mathfrak{X}} \longrightarrow \text{Der}\overline{\Lambda}$ depends on V and is defined by

$$(j_0 Z)(\omega) := \overline{L_Z \omega}, \quad Z \in \overline{\Lambda} \otimes \overline{\mathfrak{X}}, \quad \omega \in \overline{\Lambda}.$$

Finally, the homotopy $h_0 : \text{Der}\overline{\Lambda} \longrightarrow \text{Der}\overline{\Lambda}$ can be described as follows. First of all, I prove a useful

Lemma 21. *An element $\Delta \in \text{Der}\overline{\Lambda}$ can be uniquely written in the form*

$$\Delta = i_U + \overline{L}_V + \overline{L}_W, \quad (16)$$

where $U, V \in \overline{\Lambda} \otimes C\mathfrak{X}$, $W \in \overline{\Lambda} \otimes \overline{\mathfrak{X}}$, and for $X \in \Lambda(M) \otimes \mathfrak{X}(M)$ I defined $\overline{L}_X \in \text{Der}\overline{\Lambda}$ by $\overline{L}_X \omega := \overline{L_X \omega}$, $\omega \in \overline{\Lambda}$.

Proof. It is easy to check the following identity

$$[\overline{d}, \overline{L}_X] = \overline{L}_{\overline{d}X}, \quad X \in \overline{\Lambda} \otimes \mathfrak{X}(M).$$

Now, let $\Delta \in \text{Der}\overline{\Lambda}$, put

$$W := p_0 \Delta, \quad V := \Delta|_A - p_0 \Delta, \quad U := [\Delta, \overline{d}]|_A + (-)^\Delta \overline{d}W$$

and check (16). It is enough to evaluate both sides of (16) on generators. Thus, for all $f \in A$,

$$\Delta f = (V + W)f = (i_U + \overline{L}_V + \overline{L}_W)f$$

Similarly,

$$\begin{aligned} \Delta \overline{d}f &= [\Delta, \overline{d}]f + (-)^\Delta \overline{d}\Delta f \\ &= Uf - (-)^\Delta (\overline{d}W)(f) + (-)^\Delta \overline{d}(V + W)f \\ &= i_U \overline{d}f - (-)^\Delta [\overline{d}, \overline{L}_W]f + (-)^\Delta \overline{d}\overline{L}_W f + (-)^\Delta \overline{d}\overline{L}_V f \\ &= (i_U + \overline{L}_V + \overline{L}_W)\overline{d}f + \overline{d}f \end{aligned}$$

□

Then h_0 is given by

$$h_0(i_U + \bar{L}_V + \bar{L}_W) = (-)^\Delta i_V, \quad U, V \in \bar{\Lambda} \otimes C\mathfrak{X}, \quad W \in \bar{\Lambda} \otimes \bar{\mathfrak{X}}.$$

2.2. Differential Operators on a Foliated Manifold. In $\mathcal{D}(M) := \mathcal{D}(A)$, consider the left ideal $\mathcal{D}(M) \circ C\mathfrak{X}$ generated by $C\mathfrak{X}$. Denote by $\bar{\mathcal{D}}$ the quotient left $\mathcal{D}(M)$ -module $\mathcal{D}(M)/\mathcal{D}(M) \circ C\mathfrak{X}$, and by

$$\mathcal{D}(M) \ni \square \mapsto \bar{\square} \in \bar{\mathcal{D}}$$

the projection. More generally, let Q be the module of sections of a vector bundle over M . Consider the submodule $\mathcal{D}(A, Q) \circ C\mathfrak{X}$ in $\mathcal{D}(A, Q) \simeq Q \otimes \mathcal{D}(M)$, the quotient $\bar{\mathcal{D}}(A, Q) := \mathcal{D}(A, Q)/\mathcal{D}(A, Q) \circ C\mathfrak{X}$, and the projection

$$\mathcal{D}(A, Q) \ni \square \mapsto \bar{\square} \in \bar{\mathcal{D}}(A, Q). \quad (17)$$

Clearly, $\bar{\mathcal{D}}(A, Q) \simeq Q \otimes \bar{\mathcal{D}}$, and in the following I will often understand this canonical isomorphism.

The Lie algebroid $C\mathfrak{X}$ acts on $\bar{\mathcal{D}}$ as follows

$$X \cdot \bar{\square} := \overline{X \circ \square} = \overline{[X, \square]}, \quad X \in C\mathfrak{X}, \quad \square \in \mathcal{D}(M).$$

Notice that $\bar{\mathfrak{X}}$ can be understood as a submodule in $\bar{\mathcal{D}}$ and the action of $C\mathfrak{X}$ on $\bar{\mathfrak{X}}$ as the restricted action. Accordingly, the Chevalley-Eilenberg complex $(\bar{\Lambda} \otimes \bar{\mathfrak{X}}, \bar{d})$ extends to a Chevalley-Eilenberg complex $(\bar{\Lambda} \otimes \bar{\mathcal{D}}, \bar{d}_{\mathcal{D}})$ in an obvious way.

Remark 22. *The differential*

$$\bar{d}_{\mathcal{D}} : \bar{\Lambda} \otimes \bar{\mathcal{D}} \longrightarrow \bar{\Lambda} \otimes \bar{\mathcal{D}}$$

identifies with

$$\bar{d}_* : \bar{\mathcal{D}}(A, \bar{\Lambda}) \ni \bar{\square} \mapsto \bar{d}_* \bar{\square} := \overline{\bar{d} \circ \square} \in \bar{\mathcal{D}}(A, \bar{\Lambda}), \quad \square \in \mathcal{D}(A, \bar{\Lambda}).$$

Indeed, it is easy to see that both $\bar{d}_{\mathcal{D}}$ and \bar{d}_* are graded der-operators over \bar{d} . Therefore, it is enough to prove that they coincide on generators, namely, on $\bar{\mathcal{D}}$. Let $\square \in \mathcal{D}(A, \bar{\Lambda}) = \bar{\Lambda} \otimes \mathcal{D}$. Since the isomorphism $\bar{\Lambda} \otimes \mathcal{D} \longrightarrow \mathcal{D}(A, \bar{\Lambda})$ is given by $\omega \otimes \square \mapsto \omega \square$, then

$$\overline{\langle \square, X \rangle} = \langle \bar{\square}, X \rangle, \quad X \in C\mathfrak{X},$$

where I indicated with $\langle W, X \rangle$ the contraction of $W \in \bar{\Lambda}^1 \otimes Q$ with a vector fields $X \in C\mathfrak{X}$. Thus,

$$\langle \bar{d}_{\mathcal{D}} \bar{\square} | X \rangle = X \cdot \bar{\square} := \overline{X \circ \square} = \overline{\langle \bar{d} \circ \square | X \rangle} = \langle \bar{d}_* \bar{\square} | X \rangle.$$

The module $\bar{\Lambda} \otimes \bar{\mathcal{D}}$ inherits a filtration

$$\bar{\Lambda} \otimes \bar{\mathcal{D}}_0 \subset \bar{\Lambda} \otimes \bar{\mathcal{D}}_1 \subset \cdots \subset \bar{\Lambda} \otimes \bar{\mathcal{D}}_i \subset \cdots \subset \bar{\Lambda} \otimes \bar{\mathcal{D}}$$

from $\bar{\Lambda} \otimes \mathcal{D}$ and 1) the projection (17), and 2) the differential $\bar{d}_{\mathcal{D}}$, preserve this filtration. Accordingly, the graded object $\text{Gr}(\bar{\Lambda} \otimes \bar{\mathcal{D}})$ identifies with $\bar{\Lambda} \otimes S^\bullet \bar{\mathfrak{X}}$, and inherits a differential \bar{d}_S which coincides with the one in Remark 20. In particular, $(\bar{\Lambda} \otimes S^\bullet \bar{\mathfrak{X}}, \bar{d}_S)$ is a DG commutative algebra.

Consider again the complementary distribution V , and notice that, in view of commutation relations (14), $\bar{\mathcal{D}}$ is locally spanned by

$$V_{\alpha_1 \cdots \alpha_i} := \overline{V_{\alpha_1} \cdots V_{\alpha_i}}, \quad i \geq 0,$$

and they are independent generators.

Now, consider the universal enveloping DG algebra $(\mathcal{D}(\bar{\Lambda}), \delta_{\mathcal{D}})$ of the deformation complex $(\text{Der}\bar{\Lambda}, \delta_0)$. In Section 3 I show that the contraction data (p_0, j_0, h_0) for $(\text{Der}\bar{\Lambda}, \delta_0)$ over $(\bar{\Lambda} \otimes \bar{\mathfrak{X}}, \bar{d})$ extend to contraction data (p, j, h) for $(\mathcal{D}(\bar{\Lambda}), \delta_{\mathcal{D}})$ over $(\bar{\Lambda} \otimes \bar{\mathcal{D}}, \bar{d}_{\mathcal{D}})$. Here, I take only two steps in this direction. Firstly, I define the projection $p : \mathcal{D}(\bar{\Lambda}) \rightarrow \bar{\Lambda} \otimes \bar{\mathcal{D}}$, which is given by

$$p\Box := \overline{\Box|_A}, \quad \Box \in \mathcal{D}(\bar{\Lambda}),$$

and clearly extends p_0 in (15). Moreover, in view of Remark 22 and the fact that $\bar{d} : A \rightarrow \bar{\Lambda}$ does actually belong to $\mathcal{D}(A, \bar{\Lambda}) \circ C\mathfrak{X}$, $p : \mathcal{D}(\bar{\Lambda}) \rightarrow \bar{\Lambda} \otimes \bar{\mathcal{D}}$ is a cochain map. Notice that, as p_0 , p is canonical, i.e., it doesn't depend on any other structure than the distribution C . Secondly, I consider the DG object $(\mathcal{S}(\bar{\Lambda}) \simeq S_{\bar{\Lambda}}^{\bullet} \text{Der}\bar{\Lambda}, \delta_{\mathcal{S}})$ of $(\mathcal{D}(\bar{\Lambda}), \delta_{\mathcal{D}})$ and extend the contraction data for $(\text{Der}\bar{\Lambda}, \delta_0)$ over $(\bar{\Lambda} \otimes \bar{\mathfrak{X}}, \bar{d})$ to contraction data for $(\mathcal{S}(\bar{\Lambda}), \delta_{\mathcal{S}})$ over $(S_{\bar{\Lambda}}^{\bullet}(\bar{\Lambda} \otimes \bar{\mathfrak{X}}) \simeq \bar{\Lambda} \otimes S^{\bullet}\bar{\mathfrak{X}}, \bar{d}_{\mathcal{S}})$ as in Section 1.5. The next step is to construct "PBW isomorphisms"

$$\bar{\Lambda} \otimes \bar{\mathcal{D}} \approx \bar{\Lambda} \otimes S^{\bullet}\bar{\mathfrak{X}}, \quad \mathcal{D}(\bar{\Lambda}) \approx \mathcal{S}(\bar{\Lambda}).$$

This can be done exploiting an *adapted connection*. I devote the next section to the introduction of this geometric structure.

2.3. Adapted Connections. In this section C, V are complementary distributions on M . We don't require C to be involutive. The above definitions of $C\mathfrak{X}$, $\bar{\mathfrak{X}}$, $C\Lambda^1$, and $\bar{\Lambda}^1$ are still valid in the present general situation. Moreover, let

$$\mathfrak{X}(M) \ni X \mapsto CX \in C\mathfrak{X}$$

$$\Lambda^1(M) \ni \omega \mapsto \omega^C \in C\Lambda^1$$

be the projections. The pair (C, V) determines a distinguished class of connections according to the following

Definition 23. *The connection ∇ is called adapted to the pair (C, V) (or simply adapted) if*

- (1) *it restricts to $\bar{\Lambda}^1$, i.e., $\nabla_X \omega \in \bar{\Lambda}^1$ for all $X \in \mathfrak{X}(M)$ and $\omega \in \bar{\Lambda}^1$,*
- (2) *it restricts to $C\Lambda^1$, i.e., $\nabla_X \omega \in C\Lambda^1$ for all $X \in \mathfrak{X}(M)$ and $\omega \in C\Lambda^1$,*
- (3) *$\nabla_Y \omega = \overline{L_Y \omega}$ for all $Y \in \bar{\mathfrak{X}}$, $\omega \in \bar{\Lambda}^1$,*
- (4) *$\nabla_X \omega = (L_X \omega)^C$ for all $X \in C\mathfrak{X}$, $\omega \in C\Lambda^1$.*

Proposition 24. *There exist adapted connections.*

Proof. Let $\tilde{\nabla}$ be a fiducial connection. For $X \in \mathfrak{X}(M)$ and $\omega \in \Lambda^1(M)$ put

$$\nabla_X \omega = ((\tilde{\nabla}_{\bar{X}} + L_{CX})\omega^C)^C + \overline{(\tilde{\nabla}_{CX} + L_{\bar{X}})\bar{\omega}}. \quad (18)$$

The operator ∇_X is clearly a der-operator over X , i.e., $\nabla_X f\omega = X(f)\omega + f\nabla_X \omega$. Moreover, ∇_X is A -linear in X . Indeed, for $f \in A$

$$\begin{aligned} L_{CfX}\omega^C &= fL_{CX}\omega^C + df \wedge i_{CX}\omega^C = fL_{CX}\omega^C \\ L_{f\bar{X}}\bar{\omega} &= fL_{\bar{X}}\bar{\omega} + df \wedge i_{\bar{X}}\bar{\omega} = fL_{\bar{X}}\bar{\omega}. \end{aligned}$$

Thus, the correspondence $X \mapsto \nabla_X$ is a linear connection. The four properties of adapted connections are obvious. \square

Proposition 25. *Let ∇ be an adapted connection determined by a connection $\tilde{\nabla}$. Then*

$$\nabla_X Y = \overline{(\tilde{\nabla}_{\overline{X}} + L_{CX})\overline{Y}} + C(\tilde{\nabla}_{CX} + L_{\overline{X}})CY. \quad (19)$$

In particular,

- (1) ∇ restricts to $\overline{\mathfrak{X}}$, i.e., $\nabla_X Y \in \overline{\mathfrak{X}}$ for all $X \in \mathfrak{X}(M)$ and $Y \in \overline{\mathfrak{X}}$,
- (2) ∇ restricts to $C\mathfrak{X}$, i.e., $\nabla_X Y \in C\mathfrak{X}$ for all $X \in \mathfrak{X}(M)$ and $Y \in C\mathfrak{X}$,
- (3) $\nabla_Y X = C[Y, X]$, and $\nabla_X Y = \overline{[Y, X]}$ for all $Y \in \overline{\mathfrak{X}}$, $X \in C\mathfrak{X}$.

Proof. Let $X, Y \in \mathfrak{X}(M)$, and $\omega \in \Lambda^1(M)$.

$$\begin{aligned} \langle \nabla_X Y, \omega \rangle &= X \langle Y, \omega \rangle - \langle Y, \nabla_X \omega \rangle \\ &= X \langle Y, \omega \rangle - \langle \overline{Y}, (\tilde{\nabla}_{\overline{X}} + L_{CX})\omega^C \rangle - \langle CY, (\tilde{\nabla}_{CX} + L_{\overline{X}})\overline{\omega} \rangle \\ &= X \langle Y, \omega \rangle - X \langle \overline{Y}, \omega^C \rangle + \langle (\tilde{\nabla}_{\overline{X}} + L_{CX})\overline{Y}, \omega^C \rangle - X \langle CY, \overline{\omega} \rangle + \langle (\tilde{\nabla}_{CX} + L_{\overline{X}})CY, \overline{\omega} \rangle \\ &= \langle \overline{(\tilde{\nabla}_{\overline{X}} + L_{CX})\overline{Y}} + C(\tilde{\nabla}_{CX} + L_{\overline{X}})CY, \omega \rangle. \end{aligned}$$

□

It is easy to see that every adapted connection is of the form (18): for an adapted connection ∇ it is enough to put $\tilde{\nabla} = \nabla$. Indeed,

$$\begin{aligned} ((\nabla_{\overline{X}} + L_{CX})\omega^C)^C + \overline{(\nabla_{CX} + L_{\overline{X}})\overline{\omega}} &= (\nabla_{\overline{X}} + L_{CX})\omega^C + (\nabla_{CX} + L_{\overline{X}})\overline{\omega} \\ &= \nabla_X \omega^C + \nabla_X \overline{\omega} \\ &= \nabla_X \omega. \end{aligned}$$

Proposition 26. *Let ∇ be an adapted connection determined by a connection $\tilde{\nabla}$ with torsion $\tilde{T} \in \Lambda^2(M) \otimes \mathfrak{X}(M)$. The torsion T of ∇ is given by*

$$T(X, Y) = \overline{\tilde{T}(\overline{X}, \overline{Y})} + C(\tilde{T}(CX, CY)) - \overline{[CX, CY]} - C[\overline{X}, \overline{Y}], \quad X, Y \in \mathfrak{X}.$$

Proof. Compute

$$\begin{aligned} T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= \overline{(\tilde{\nabla}_{\overline{X}} + L_{CX})\overline{Y}} + C(\tilde{\nabla}_{CX} + L_{\overline{X}})CY \\ &\quad - \overline{(\tilde{\nabla}_{\overline{Y}} + L_{CY})\overline{X}} - C(\tilde{\nabla}_{CY} + L_{\overline{Y}})CX - [X, Y] \\ &= \overline{\tilde{T}(\overline{X}, \overline{Y})} + C\tilde{T}(CX, CY) + \overline{[\overline{X}, \overline{Y}]} + C[CX, CY] \\ &\quad + \overline{[CX, \overline{Y}]} + C[\overline{X}, CY] + \overline{[\overline{X}, CY]} + C[CX, \overline{Y}] - [X, Y] \\ &= \overline{\tilde{T}(\overline{X}, \overline{Y})} + C\tilde{T}(CX, CY) - \overline{[CX, CY]} - C[\overline{X}, \overline{Y}]. \end{aligned}$$

□

Corollary 27. *A torsion-free adapted connection exists iff both C and V are involutive.*

Proof. If both C and V are involutive, an adapted connection determined by a torsion-free connection is torsion-free as well. Conversely, let the adapted connection ∇ determined by a connection $\tilde{\nabla}$ be torsion-free. Then, for $X, Y \in \mathfrak{X}$,

$$0 = T(X, Y) = \overline{\tilde{T}(X, Y)} - C[X, Y] \implies C[X, Y] = 0.$$

Similarly, for $X, Y \in C\mathfrak{X}$. □

Definition 28. *An adapted connection ∇ is called torsion-quasi-free if*

$$T(X, Y) = -\overline{[CX, CY]} - C[\overline{X}, \overline{Y}].$$

Corollary 29. *There exist torsion-quasi-free adapted connections.*

Proof. The adapted connection determined by a torsion-free connection is torsion-quasi-free. □

Now, suppose that C is involutive and let ∇ be a torsion-quasi-free adapted connection. Let T be the torsion of ∇ . Then, clearly,

- (1) ∇ extends the Bott connection,
- (2) T coincides with the canonical metaplectic structure in V , up to a sign,
- (3) In view of (14)

$$T(V_\alpha, V_\beta) = -R_{\alpha\beta}^i \partial_i. \quad (20)$$

2.4. Two PBW Isomorphisms. Now, let C be again an involutive distribution on M , and ∇ a connection in $\Lambda^1(M)$ adapted to the pair (C, V) and torsion-quasi-free. The connection ∇ determines two PBW type isomorphisms (see [14] for a similar result)

$$\underline{\text{PBW}} : \overline{\Lambda} \otimes \overline{\mathcal{D}} \approx \overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}, \quad \text{PBW} : \mathcal{D}(\overline{\Lambda}) \approx \mathcal{S}(\overline{\Lambda})$$

as follows. For $\omega \in \overline{\Lambda}$ and $P \in S^\bullet \overline{\mathfrak{X}}$, put

$$\underline{\text{PBW}}(\omega \otimes P) := \omega \otimes \overline{\nabla_P},$$

where ∇_P is defined as in Example 19. To define PBW notice, first of all, that every derivation $\Delta \in \text{Der} \overline{\Lambda}$ can be uniquely written in the form

$$\Delta = i_W + \nabla_V + \overline{L}_Z, \quad (21)$$

$W, V \in \overline{\Lambda} \otimes C\mathfrak{X}$, $Z \in \overline{\Lambda} \otimes \overline{\mathfrak{X}}$, where ∇_V is defined as in Example 19. Indeed, let

$$\Delta = i_U + \overline{L}_V + \overline{L}_Z,$$

$U, V \in \overline{\Lambda} \otimes C\mathfrak{X}$, $Z \in \overline{\Lambda} \otimes \overline{\mathfrak{X}}$. Now, since ∇ is adapted, and torsion-quasi-free, then $\overline{L}_V = \nabla_V + i_{\nabla V}$, with $\nabla V \in \overline{\Lambda} \otimes C\mathfrak{X}$. Thus (21) holds simply putting $W := U + \nabla V$. Clearly, the correspondence

$$\overline{\Lambda} \otimes C\mathfrak{X}[1] \oplus \overline{\Lambda} \otimes C\mathfrak{X} \oplus \overline{\Lambda} \otimes \overline{\mathfrak{X}} \ni (W, V, Z) \mapsto i_W + \nabla_V + \overline{L}_Z \in \text{Der} \overline{\Lambda}$$

is an isomorphism of $\overline{\Lambda}$ -module, so that

$$\mathcal{S}(\overline{\Lambda}) \simeq \overline{\Lambda} \otimes \Lambda^\bullet C\mathfrak{X} \otimes S^\bullet C\mathfrak{X} \otimes S^\bullet \overline{\mathfrak{X}}$$

and $p_0 : \mathcal{S}(\overline{\Lambda}) \rightarrow \overline{\Lambda} \otimes S^\bullet \overline{\mathfrak{X}}$ is the obvious projection. Moreover, let

$$\Sigma = \omega \otimes X_1 \wedge \cdots \wedge X_k \otimes P \otimes Q \in \mathcal{S}(\overline{\Lambda}),$$

put

$$\text{PBW}(\Sigma) := \omega i_{X_1} \cdots i_{X_k} \nabla_{P \odot Q} \in \mathcal{D}(\overline{\Lambda}).$$

Remark 30. In general, the isomorphisms $\underline{\text{PBW}}$, and PBW are not compatible with projections p, p_0 . However, if one uses them to induce an injection $j_0^\sim : \bar{\Lambda} \otimes \bar{\mathcal{D}} \rightarrow \mathcal{D}(\bar{\Lambda})$, from the injection $j_0 : \bar{\Lambda} \otimes S^\bullet \bar{\mathcal{X}} \rightarrow \mathcal{S}(\bar{\Lambda})$, then the former is a right inverse of p . Indeed, clearly

$$\underline{\text{PBW}} = p \circ \text{PBW} \circ j_0,$$

so that, for $\Sigma \in \bar{\Lambda} \otimes S^i \bar{\mathcal{X}}$

$$\begin{aligned} (p \circ j_0^\sim \circ \underline{\text{PBW}})\Sigma &= (p \circ j_0^\sim \circ p \circ \text{PBW} \circ j_0)\Sigma \\ &= (p \circ \text{PBW} \circ j_0 \circ \sigma_i \circ p \circ \text{PBW} \circ j_0)\Sigma \\ &= (p \circ \text{PBW} \circ j_0 \circ p_0 \circ \sigma_i \circ \text{PBW} \circ j_0)\Sigma \\ &= (p \circ \text{PBW} \circ j_0 \circ p_0 \circ j_0)\Sigma \\ &= (p \circ \text{PBW} \circ j_0)\Sigma \\ &= \underline{\text{PBW}}(\Sigma). \end{aligned}$$

In the following, I will often understand isomorphisms $\underline{\text{PBW}}$ and PBW .

3. THE A_∞ -ALGEBRA OF A FOLIATION

Let M be a smooth manifold and C an involutive distribution on it. Summarizing results obtained so far, a complementary distribution V and a torsion-quasi-free adapted connection ∇ determine

- (1) Contraction data (p_0, j_0, h_0) for $(\mathcal{S}(\bar{\Lambda}), \delta_{\mathcal{S}})$ over $(\bar{\Lambda} \otimes S^\bullet \bar{\mathcal{X}}, \bar{d}_{\mathcal{S}})$,
- (2) PBW type isomorphisms $\bar{\Lambda} \otimes \bar{\mathcal{D}} \approx \bar{\Lambda} \otimes S^\bullet \bar{\mathcal{X}}$, $\mathcal{D}(\bar{\Lambda}) \approx \mathcal{S}(\bar{\Lambda})$,

Notice that, actually, i) p_0 is independent of the supplementary geometric data V and ∇ , and 2) j_0, h_0 do only depend on V .

Now, put $t = \delta_{\mathcal{S}} - \delta_{\mathcal{D}} : \mathcal{D}(\bar{\Lambda}) \rightarrow \mathcal{D}(\bar{\Lambda})$. Perturbation Lemma determine a “new” differential $\bar{d}_t : \bar{\Lambda} \otimes \bar{\mathcal{D}} \rightarrow \bar{\Lambda} \otimes \bar{\mathcal{D}}$ and contraction data (p_t, j_t, h_t) for $(\mathcal{D}(\bar{\Lambda}), \delta_{\mathcal{D}})$ over $(\bar{\Lambda} \otimes \bar{\mathcal{D}}, \bar{d}_t)$, given by formulas (6),(7),(8). In its turn, the Homotopy Transfer Theorem determine an A_∞ -algebra structure in $(\bar{\Lambda} \otimes \bar{\mathcal{D}}, \bar{d}_t)$. Before giving more detail about this structures, I remark that p_t is actually independent of V and ∇ and coincides with the canonical projection $p : \mathcal{D}(\bar{\Lambda}) \rightarrow \bar{\Lambda} \otimes \bar{\mathcal{D}}$. To show this, first notice that $pj_0 = \text{id}$ and $ph_0 = 0$ (the first identity is discussed in Remark 30, while the second one is immediate from the definitions of h_0 and p). It follows that $pj_t = \text{id}$ and $ph_t = 0$. Now, let $\square \in \mathcal{D}(\bar{\Lambda})$. Then

$$0 = p[h_t, \delta]\square = p(\text{id} - j_t p_t)\square = (p - p_t)\square.$$

As an immediate consequence, \bar{d}_t is also independent of V and ∇ and coincide with the canonical differential $\bar{d}_{\mathcal{D}} : \bar{\Lambda} \otimes \bar{\mathcal{D}} \rightarrow \bar{\Lambda} \otimes \bar{\mathcal{D}}$. In the following, I put $j := j_t$ and $h := h_t$.

I am finally in the position to furnish few details about the (strict unital) A_∞ -algebra structure $\{\alpha_k, k \in \mathbb{N}\}$ on $\bar{\Lambda} \otimes \bar{\mathcal{D}}$. To this aim, notice that the isomorphism $\text{PBW} : \mathcal{D}(\bar{\Lambda}) \approx \bar{\mathcal{S}}_\bullet$ (resp., $\underline{\text{PBW}} : \bar{\Lambda} \otimes \bar{\mathcal{D}} \approx \bar{\Lambda} \otimes S^\bullet \bar{\mathcal{X}}$) determines a new grading in $\mathcal{D}(\bar{\Lambda})$ (resp., $\bar{\Lambda} \otimes \bar{\mathcal{D}}$), which I call the *order* and is given by the decomposition $\mathcal{S}(\bar{\Lambda}) = \bigoplus_k \mathcal{S}_k \bar{\Lambda}$ (resp., $\bar{\Lambda} \otimes S^\bullet \bar{\mathcal{X}} = \bigoplus_k \bar{\Lambda} \otimes S^k \bar{\mathcal{X}}$). Every map ϕ of the spaces $\mathcal{D}(\bar{\Lambda})$ and $\bar{\Lambda} \otimes \bar{\mathcal{D}}$ have its homogenous components with respect to the order. I denote by $\phi^{[i]}$ the i -th one, and by $\mathcal{O}(i)$ a generic (no better specified) object of order no higher than i , e.g.,

$$\delta_{\mathcal{D}} = \delta_{\mathcal{S}} + \mathcal{O}(-1), \quad \bar{d}_{\mathcal{D}} = \bar{d}_{\mathcal{S}} + \mathcal{O}(-1), \quad p = p_0 + \mathcal{O}(-1), \quad h = h_0 + \mathcal{O}(-1).$$

Similarly,

$$t = t^{[-1]} + \mathcal{O}(-2),$$

and

$$j = j_0 + h_0 t^{[-1]} j_0 + \mathcal{O}(-2). \quad (22)$$

I will not need to compute $t^{[-1]}$. Finally, the composition \circ of differential operators in $\mathcal{D}(\bar{\Lambda})$, decomposes as

$$\circ = \odot + \circledast + \mathcal{O}(-2),$$

where I put $\circledast := \circ^{[-1]}$.

Notice that, in view of the above decompositions of $\delta_{\mathcal{D}}, \bar{d}_{\mathcal{D}}$ and the contraction data (p, j, h) , the k -th Poisson bracket in $\bar{\Lambda} \otimes S^\bullet \bar{\mathfrak{X}}$ is the skew-symmetrization $\mathbf{A}\alpha_k^{[1-k]}$ of $\alpha_k^{[1-k]}$. In particular, the skew-symmetrization of $\mathbf{A}\alpha_k^{[1-k]}$ vanishes for $k > 3$ [24]. My next aim is twofold:

- (1) proving that $\alpha_k^{[1-k]}$ is the highest order component of α_k , i.e., $\alpha_k = \alpha_k^{[1-k]} + \mathcal{O}(-k)$, for $k \neq 2$,
- (2) “computing” $\alpha_k^{[1-k]}$ and, in particular, showing that it is zero for $k > 3$.

Notice that the first claim states that the order of $\alpha_k(\square_1, \dots, \square_k)$ is no higher than $1 - k + \sum_i \ell_i$ for $\square_i = \mathcal{O}(\ell_i)$, $i = 1, \dots, k$. The claim that $\alpha_k^{[1-k]} = 0$ for $k > 3$, can be interpreted as a further motivation why the LR_∞ -algebra structure in $\bar{\Lambda} \otimes \bar{\mathfrak{X}}$ presents just one higher homotopy [24]. In order to reach my aim, I first prove a

Lemma 31. *The order -1 component of the projection p vanishes, i.e., $p^{[-1]} = 0$ (so that $p = p_0 + \mathcal{O}(-2)$).*

Proof. Let $\square \in \mathcal{D}(\bar{\Lambda})$ be of the order H . Then, \square is locally of the form

$$\square = \sum_{k+\ell+m=H} A^{i_1 \dots i_k | j_1 \dots j_\ell | \alpha_1 \dots \alpha_m} I_{i_1 \dots i_k} \nabla_{j_1} \dots \nabla_{j_\ell} \nabla_{\alpha_1} \dots \nabla_{\alpha_m}$$

where $I_{i_1 \dots i_k} := i_{\partial_{i_1}} \dots i_{\partial_{i_k}}$, and the A ’s are components of a (contravariant) tensor. The A ’s are skew-symmetric in the i ’s, symmetric in the j ’s and symmetric in the α ’s. Compute

$$p\square = \sum_{\ell+m=H} A^{\partial | j_1 \dots j_\ell | \alpha_1 \dots \alpha_m} \overline{\nabla_{j_1} \dots \nabla_{j_\ell} \nabla_{\alpha_1} \dots \nabla_{\alpha_m}}$$

Clearly,

$$p^{[0]}\square = (p\square)^{[H]} = p_0\square = A^{\partial | \partial | \alpha_1 \dots \alpha_H} V_{\alpha_1 \dots \alpha_H}.$$

Now, let $\ell > 0$,

$$\begin{aligned} & A^{\partial | j_1 \dots j_\ell | \alpha_1 \dots \alpha_m} \overline{\nabla_{j_1} \dots \nabla_{j_\ell} \nabla_{\alpha_1} \dots \nabla_{\alpha_m}} \\ &= A^{\partial | j_1 \dots j_\ell | \alpha_1 \dots \alpha_m} \overline{\nabla_{j_1} \dots \nabla_{j_{\ell-1}} [\nabla_{j_\ell}, \nabla_{\alpha_1} \dots \nabla_{\alpha_m}]} \\ &= A^{\partial | j_1 \dots j_\ell | \alpha_1 \dots \alpha_m} \sum_{r \leq m} \overline{\nabla_{j_1} \dots \nabla_{j_{\ell-1}} \nabla_{\alpha_1} \dots [\nabla_{j_\ell}, \nabla_{\alpha_r}] \dots \nabla_{\alpha_m}}. \end{aligned}$$

Since ∇ is adapted and torsion-quasi-free

$$[\nabla_i, \nabla_\alpha] \lambda_{\beta_1 \dots \beta_t} = \sum_{s \leq t} R_{i\alpha\beta_s}^\nabla \lambda_{\beta_1 \dots \beta_{s-1} \beta_{s+1} \dots \beta_t}, \quad (23)$$

for all covariant tensors λ locally of the form

$$\lambda = \lambda_{\beta_1 \dots \beta_t} du^{\beta_1} \otimes \dots \otimes du^{\beta_t}.$$

In (23) R^∇ is the curvature tensor of ∇ . It follows from (23), that $[\nabla_i, \nabla_\alpha] = \mathcal{O}(0)$, and

$$(\overline{\nabla_{(j_1} \dots \nabla_{j_\ell)} \nabla_{(\alpha_1} \dots \nabla_{\alpha_m)}}) \in \mathcal{O}(\ell + m - 2)$$

for all ℓ, m . I conclude that

$$p\Box = p_0\Box + \mathcal{O}(H - 2).$$

□

The following proposition is a corollary of the above lemma, and the side conditions $ph = 0$, $hj = 0$, $h^2 = 0$.

Proposition 32.

$$\begin{aligned} \gamma_k &= \mathcal{O}(1 - k), & k &\geq 1 \\ \beta_k &= \mathcal{O}(2 - k), & k &\geq 2 \\ \alpha_k &= \mathcal{O}(1 - k), & k &\geq 3 \end{aligned}$$

while

$$\alpha_2 = \mathcal{O}(0).$$

Moreover, the highest order component $\alpha_k^{[1-k]}$ of α_k can be computed iteratively via formulas

$$\begin{aligned} \varepsilon_k(\Box_1, \dots, \Box_k) &:= - \sum_{\ell+m=k} (-)^{a(\ell, m, \Box)} \gamma_\ell^{[1-\ell]}(\Box_1, \dots, \Box_\ell) \otimes \gamma_m^{[1-m]}(\Box_{\ell+1}, \dots, \Box_{\ell+m}) \\ \gamma_k^{[1-k]} &= h_0 \varepsilon_k, \\ \alpha_k^{[1-k]} &= p_0 \varepsilon_k, \end{aligned}$$

$\Box = (\Box_1, \dots, \Box_k) \in (\overline{\Lambda} \otimes \overline{\mathcal{D}})^k$, being a k -tuple of homogeneous elements of given orders, $k \geq 2$.

Proof. The two parts of the proposition can be checked simultaneously by induction on k . Indeed, $\gamma_1 = -j = -j_0 + \mathcal{O}(-1)$, $\beta_2 = j(-) \circ j(-) = j_0(- \odot -) + \mathcal{O}(-1)$, and $\alpha_2 = (- \odot -) + \mathcal{O}(-1)$ (where I used that p_0 preserves the product \odot). Moreover,

$$\gamma_2 = h\beta_2 = h(j(-) \circ j(-)),$$

so that

$$\gamma_2^{[0]} = h_0 j_0(- \odot -) = 0.$$

Thus, compute

$$\begin{aligned} \gamma_2^{[-1]} &= h^{[-1]} j_0(- \odot -) + h_0(j^{[-1]}(-) \odot j_0(-) + j_0(-) \otimes j_0(-) + j_0(-) \odot j^{[-1]}(-)) \\ &= h_0(j_0(-) \otimes j_0(-)) \end{aligned}$$

where I used formulas (8), (22). Now,

$$\beta_k = \sum_{\ell+m=k} (-)^{\ell-1} \gamma_\ell(-) \circ \gamma_m(-)$$

with $\gamma_\ell = \mathcal{O}(1 - \ell)$ and $\gamma_m = \mathcal{O}(1 - m)$ by induction hypothesis. Therefore, it is immediately seen that $\beta_k = \mathcal{O}(2 - k)$, and

$$\beta_k^{[2-k]} = \sum_{\ell+m=k} (-)^{\ell-1} \gamma_\ell^{[1-\ell]}(-) \odot \gamma_m^{[1-m]}(-),$$

so that

$$\gamma_k = h\beta_k = \mathcal{O}(2 - k).$$

But

$$\begin{aligned} \gamma_k^{[2-k]} &= \sum_{\ell+m=k} (-)^{\ell-1} h_0(\gamma_\ell^{[1-\ell]}(-) \odot \gamma_m^{[1-m]}(-)) \\ &= \sum_{\ell+m=k} (-)^{\ell-1} h_0(h_0\varepsilon_\ell(-) \odot h_0\varepsilon_m(-)) \\ &= 0. \end{aligned}$$

Now, compute

$$\beta_k^{[1-k]} = \sum_{\ell+m=k} (-)^{\ell-1} (\gamma_\ell^{[-\ell]}(-) \odot \gamma_m^{[1-m]}(-) + \gamma_\ell^{[1-\ell]}(-) \otimes \gamma_m^{[1-m]}(-) + \gamma_\ell^{[1-\ell]}(-) \odot \gamma_m^{[-m]}(-)),$$

and

$$\gamma_k^{[1-k]} = h^{[-1]} \beta_k^{[2-k]} + h_0 \beta_k^{[1-k]} = h_0 \beta_k^{[1-k]} = h_0 \varepsilon_k,$$

where I used (8).

Finally, compute

$$\alpha_k = p\beta_k = \mathcal{O}(2 - k).$$

But

$$\alpha_k^{[2-k]} = p_0 \beta_k^{[2-k]} = \sum_{\ell+m=k} (-)^{\ell-1} p_0 \gamma_\ell^{[1-\ell]}(-) \odot p_0 \gamma_m^{[1-m]}(-) = 0,$$

where I used the side condition $p_0 h_0 = 0$, and

$$\alpha_k^{[1-k]} = p^{[-1]} \beta_k^{[2-k]} + p_0 \beta_k^{[1-k]} = p_0 \beta_k^{[1-k]} = p_0 \varepsilon_k,$$

where I used the above lemma and the side condition $p_0 h_0 = 0$ again. \square

In view of the above proposition, a formula for \otimes is enough to get inductive formulas for the $\alpha_k^{[1-k]}$'s. These formulas, which I compute in the proof of the next lemma, actually show that $\alpha_k^{[1-k]} = 0$ for $k > 3$, as announced.

Now on put

$$\mathcal{S}_{i,j,\ell} := \overline{\Lambda} \otimes \Lambda^i C\mathfrak{X} \otimes S^j C\mathfrak{X} \otimes S^\ell \overline{\mathfrak{X}} \subset \mathcal{S}(\overline{\Lambda})$$

Lemma 33. *Let $\square_1 \in \mathcal{S}_{r,0,\ell}$ and $\square_2 \in \mathcal{S}_{s,0,m}$, then*

$$\begin{aligned} \square_1 \otimes \square_2 &\in \mathcal{S}_{r+s,1,\ell+m-2} + \mathcal{S}_{r+s,0,\ell+m-1} + \mathcal{S}_{r+s-1,0,\ell+m} \\ h_0(\square_1 \otimes \square_2) &\in \mathcal{S}_{r+s+1,0,m+\ell-2} \\ p_0(\square_1 \otimes \square_2) &\in \begin{cases} \overline{\Lambda} \otimes S^{\ell+m-1} \overline{\mathfrak{X}} & \text{if } r+s=0 \\ \overline{\Lambda} \otimes S^{\ell+m} \overline{\mathfrak{X}} & \text{if } r+s=1 \\ 0 & \text{if } r+s>1 \end{cases} \end{aligned}$$

Proof. The operators \square_1 and \square_2 are locally of the form

$$\begin{aligned}\square_1 &= \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_\ell} I_{i_1 \dots i_r} \nabla_{\alpha_1} \dots \nabla_{\alpha_\ell} \\ \square_2 &= \Psi^{j_1 \dots j_s | \beta_1 \dots \beta_m} I_{j_1 \dots j_s} \nabla_{\beta_1} \dots \nabla_{\beta_m}.\end{aligned}$$

Then

$$\begin{aligned}\square_1 \circ \square_2 &= \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_\ell} \Psi^{j_1 \dots j_s | \beta_1 \dots \beta_m} I_{i_1 \dots i_r j_1 \dots j_s} \nabla_{(\alpha_1 \dots \nabla_{\alpha_\ell} \nabla_{\beta_1} \dots \nabla_{\beta_m})} \\ &\quad + \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_\ell} \Psi^{j_1 \dots j_s | \beta_1 \dots \beta_m} I_{i_1 \dots i_r j_1 \dots j_s} (\nabla_{(\alpha_1 \dots \nabla_{\alpha_\ell})} \nabla_{(\beta_1 \dots \nabla_{\beta_m})})^{[\ell+m-1]} \\ &\quad + \ell \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_\ell} \nabla_{\alpha_1} \Psi^{j_1 \dots j_s | \beta_1 \dots \beta_m} I_{i_1 \dots i_r j_1 \dots j_s} \nabla_{(\alpha_2 \dots \nabla_{\alpha_\ell} \nabla_{\beta_1} \dots \nabla_{\beta_m})} \\ &\quad + r \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_\ell} I_{i_1} \Psi^{j_1 \dots j_s | \beta_1 \dots \beta_m} I_{i_2 \dots i_r j_1 \dots j_s} \nabla_{(\alpha_1 \dots \nabla_{\alpha_\ell} \nabla_{\beta_1} \dots \nabla_{\beta_m})} + O(\ell + m - 2).\end{aligned}$$

It remains to compute

$$(\nabla_{(\alpha_1 \dots \nabla_{\alpha_\ell})} \nabla_{(\beta_1 \dots \nabla_{\beta_m})})^{[\ell+m-1]}.$$

Let $A^{\alpha_1 \dots \alpha_\ell | \beta_1 \dots \beta_m}$ be symmetric in the α 's and the β 's separately. Then

$$\begin{aligned}A^{\alpha_1 \dots \alpha_\ell | \beta_1 \dots \beta_m} (\nabla_{(\alpha_1 \dots \nabla_{\alpha_\ell})} \nabla_{(\beta_1 \dots \nabla_{\beta_m})})^{[\ell+m-1]} \\ = A^{\alpha_1 \dots \alpha_\ell | \beta_1 \dots \beta_m} (\nabla_{\alpha_1} \dots \nabla_{\alpha_\ell} \nabla_{\beta_1} \dots \nabla_{\beta_m})^{[\ell+m-1]} \\ = \frac{m}{m+1} A^{\alpha_1 \dots \alpha_{\ell-1} | \beta_1 \dots \beta_{m-1}} R_{\alpha_\ell}^i \nabla_i \nabla_{(\alpha_1 \dots \nabla_{\alpha_{\ell-1}} \nabla_{\beta_1} \dots \nabla_{\beta_{m-1}})}.\end{aligned}$$

I conclude that

$$\begin{aligned}\square_1 \circ \square_2 &= \frac{m}{m+1} R_{\alpha_\ell}^i \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_{\ell-1}} \Psi^{j_1 \dots j_s | \beta_1 \dots \beta_{\ell+m-2}} i_{i_1 \dots i_r j_1 \dots j_s} \nabla_i \nabla_{(\alpha_1 \dots \nabla_{\alpha_{\ell+m-2}})} \\ &\quad + \ell \Phi^{i_1 \dots i_r | \alpha_1 \dots \alpha_{\ell-1}} \nabla_{\alpha_\ell} \Psi^{j_1 \dots j_s | \alpha_\ell \dots \alpha_{\ell+m-1}} i_{i_1 \dots i_r j_1 \dots j_s} \nabla_{(\alpha_1 \dots \nabla_{\alpha_{\ell+m-1}})} \\ &\quad + r \Phi^{i_1 \dots i_{r-1} | \alpha_1 \dots \alpha_\ell} i_{i_r} \Psi^{j_1 \dots j_s | \alpha_{\ell+1} \dots \alpha_{\ell+m}} i_{i_1 \dots i_{r-1} j_1 \dots j_s} \nabla_{(\alpha_1 \dots \nabla_{\alpha_{\ell+m}})}\end{aligned}\tag{24}$$

□

Corollary 34. Let $\square_1, \dots, \square_k \in \overline{\Lambda} \otimes \overline{\mathcal{D}}$ with \square_i being of the order ℓ_i . Put $\ell := \ell_1 + \dots + \ell_k$. Then

$$\begin{aligned}\gamma_k^{[1-k]}(\square_1, \dots, \square_k) &\in \mathcal{S}_{k-1,0,\ell-2k+2}, \\ \varepsilon_k(\square_1, \dots, \square_k) &\in \mathcal{S}_{k-2,1,\ell-2k+2} + \mathcal{S}_{k-2,0,\ell-2k+3} + \mathcal{S}_{k-3,0,\ell-2k+4}, \\ \alpha_k^{[1-k]}(\square_1, \dots, \square_k) &\in \begin{cases} \overline{\Lambda} \otimes S^{\ell-1} \overline{\mathfrak{X}} & \text{if } k = 2 \\ \overline{\Lambda} \otimes S^{\ell-2} \overline{\mathfrak{X}} & \text{if } k = 3 \\ 0 & \text{if } k > 3 \end{cases},\end{aligned}$$

$k > 1$.

Proof. Immediate, by induction on k . □

Now, compute $\alpha_k^{[1-k]}$, for $k = 1, 2, 3$. Let $\square_1, \square_2, \square_3 \in \overline{\Lambda} \otimes \overline{\mathcal{D}}$ be locally given by

$$\square_i = \Phi_i^{\alpha_1 \dots \alpha_r} V_{\alpha_1 \dots \alpha_r}, \quad i = 1, 2, 3.$$

First of all,

$$\alpha_2^{[0]}(\square_1, \square_2) = \Phi_1^{\alpha_1 \dots \alpha_r} \Phi_2^{\alpha_{r+1} \dots \alpha_{r+s}} V_{\alpha_1 \dots \alpha_{r+s}}.$$

Moreover, using Formula (24), it is easy to see that

$$\begin{aligned}\alpha_2^{[-1]}(\square_1, \square_2) &= r\Phi_1^{\alpha\alpha_1\cdots\alpha_{r-1}}\nabla_\alpha\Phi_2^{\alpha_r\cdots\alpha_{r+s-1}}V_{\alpha_1\cdots\alpha_{r+s-1}} \\ \alpha_3^{[-2]}(\square_1, \square_2, \square_3) &= \frac{2t}{t+1}R_{\alpha\beta}^i\Phi_1^{\alpha\alpha_1\cdots\alpha_{r-1}}\Phi_2^{\beta\alpha_r\cdots\alpha_{r+s-2}}I_i\Phi_3^{\alpha_{r+s-1}\cdots\alpha_{r+s+t-2}}V_{\alpha_1\cdots\alpha_{r+s+t-2}}\end{aligned}$$

which are duly consistent with formulas in [24].

Remark 35. Notice that the natural $\mathcal{D}(\overline{\Lambda})$ -module structure in $\overline{\Lambda}$ can be transferred along the contraction data (p, j, h) as well. Indeed, $\overline{\Lambda}$ is actually a DG module over $\mathcal{D}(\overline{\Lambda})$ with differential $\bar{d} : \overline{\Lambda} \rightarrow \overline{\Lambda}$. Moreover, this DG module structure (and the DG algebra structure in $\mathcal{D}(\overline{\Lambda})$) can be encoded in a DG algebra structure in $\mathcal{D}(\overline{\Lambda}) \oplus \overline{\Lambda}$ given by

$$(\square_1, \omega_1)(\square_2, \omega_2) := (\square_1 \circ \square_2, \square_1\omega_2), \quad (\square_i, \omega_i) \in \mathcal{D}(\overline{\Lambda}) \oplus \overline{\Lambda}, \quad i = 1, 2,$$

with differential $\delta^\oplus := \delta_{\mathcal{D}} \oplus \bar{d}$. Similarly, consider the complex $(\overline{\Lambda} \otimes \overline{\mathcal{D}} \oplus \overline{\Lambda}, \bar{d}^\oplus)$ where $\bar{d}^\oplus := \bar{d}_{\mathcal{D}} \oplus \bar{d}$. There are obvious retraction data $(p^\oplus, j^\oplus, h^\oplus)$ of $(\mathcal{D}(\overline{\Lambda}) \oplus \overline{\Lambda}, \delta^\oplus)$ over $(\overline{\Lambda} \otimes \overline{\mathcal{D}} \oplus \overline{\Lambda}, \bar{d}^\oplus)$. Namely,

$$p^\oplus := p \oplus \text{id}, \quad j^\oplus := j \oplus \text{id}, \quad h^\oplus := h \oplus 0.$$

Accordingly, there is a A_∞ -algebra structure $\{\alpha_k^\oplus, k \in \mathbb{N}\}$ in $\overline{\Lambda} \otimes \overline{\mathcal{D}} \oplus \overline{\Lambda}$. By construction,

$$\alpha_k^\oplus((\square_1, \omega_1), \dots, (\square_k, \omega_k)) = \alpha_k(\square_1, \dots, \square_k) + \alpha_k^\oplus(\square_1, \dots, \square_{k-1}, \omega_k),$$

$(\square_i, \omega_i) \in \mathcal{D}(\overline{\Lambda}) \oplus \overline{\Lambda}$, $i = 1, \dots, k$. Therefore, if one put

$$\mu_k(\square_1, \dots, \square_{k-1}|\omega) := \alpha_k^\oplus(\square_1, \dots, \square_{k-1}, \omega),$$

then $\{\mu_k, k \in \mathbb{N}\}$ is an A_∞ -module structure in $\overline{\Lambda}$, over the A_∞ -algebra $\overline{\Lambda} \otimes \overline{\mathcal{D}}$. It is easy to see that, since $h^\oplus \rho = 0$ for all $\rho \in \overline{\Lambda}$, then the μ 's are simply given by

$$\begin{aligned}\mu_1 &= \bar{d} \\ \mu_k(\square_1, \dots, \square_{k-1}|\omega) &= (-)^{k-1}\gamma_k(\square_1, \dots, \square_{k-1})\omega, \quad k \geq 2.\end{aligned}$$

CONCLUSIONS

I proved that the LR_∞ -algebra $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \mathcal{L})$ of a foliation [24] can be actually extended in a natural way to an A_∞ -algebra $(\overline{\Lambda} \otimes \overline{\mathcal{D}}, \mathcal{A})$ of longitudinal form-valued normal differential operators. This can be done via purely geometric data, namely a distribution complementary to the characteristic distribution and a connection (of a suitable kind). Notice that $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \mathcal{L})$ can be interpreted (to some extent) as the (derived) Lie-Rinehart algebra of vector fields on the space \mathbb{M} of integral manifolds. Similarly, it is natural to interpret $(\overline{\Lambda} \otimes \overline{\mathcal{D}}, \mathcal{A})$ as the (derived) associative algebra of differential operators on \mathbb{M} . In this respect, it is tempting to conjecture that $(\overline{\Lambda} \otimes \overline{\mathcal{D}}, \mathcal{A})$ is a universal enveloping SH algebra of $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \mathcal{L})$. However, the theory of universal enveloping of LR_∞ -algebras (or L_∞ -algebroids) is not yet available and developing this research line goes beyond the scopes of this paper. Here, I just notice that $(\overline{\Lambda} \otimes \overline{\mathcal{D}}, \mathcal{A})$ is indeed a (possibly non universal) *enveloping SH algebra* of $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \mathcal{L})$ in the following sense. The inclusion $\overline{\Lambda} \otimes \overline{\mathfrak{X}} \rightarrow \overline{\Lambda} \otimes \overline{\mathcal{D}}$ can be trivially extended to a morphism

$J : \overline{\Lambda} \otimes \overline{\mathfrak{X}} \longrightarrow \overline{\Lambda} \otimes \overline{\mathcal{D}}$ of the L_∞ -algebra $(\overline{\Lambda} \otimes \overline{\mathfrak{X}}, \mathcal{L})$ and the L_∞ -algebra obtained by skew-symmetrization of operations in \mathcal{A} , simply putting $J_k = 0$ for $k > 1$. Then, it is easy to see, using the explicit formulas for brackets in \mathcal{L} [24], that

$$\nu_k(Z_1, \dots, Z_{k-1}|\omega) = (A\alpha_k)(Z_1, \dots, Z_{k-1}, \omega), \quad \omega \in \overline{\Lambda}, \quad Z_i \in \overline{\Lambda} \otimes \overline{\mathfrak{X}},$$

$i = 1, \dots, k-1$, which specializes (5) to the present simple case where j is an inclusion and $J_k = 0$ for $k > 1$.

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